Some recent progresses in group connectivity and modulo orientations of graphs

Hong-Jian Lai

In honor of Professor Yanpei LIU

Notation

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- Theorem (Tutte) If G has a k-NZF, then G has a $(k+1)$-NZF.
- Theorem (Tutte) A graph G has an A-NZF if and only if G has an $|A|$-NZF.

The Nowhere Zero Flow Conjectures

- Fact: If some orientation $D(G)$ has an A-NZF or a k-NZF, then for any orientation of G also has the same property, and so having an A-NZF or a k-NZF is independent of the choice of the orientation.

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- Tutte's 3-flow Conjecture If $\kappa'(G) \geq 4$, then G has a 3-NZF.

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- These conjectures are theorems when restricted to planar graphs (need 4 Color Theorem for the 4-flow conjecture).

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- The corresponding nonhomogeneous problem is, for a given vector $b : V(G) \mapsto A$, determine if there is a nowhere zero solution f to the system $\partial f = b$.

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- as for every arc (u, v), both $f(u, v)$ and $-f(u, v)$ occur in the summation exactly once.

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- Conjecture 2 (Jaeger et al, JCTB 1992) If $\kappa'(G) \geq 5$, then for any abelian group A with $|A| \geq 3$, G is A-connected.
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- This orientation D is called a mod 3-orientation of G.

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- M_k:= family of all graphs admitting a mod k-orientation.

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- Proposition (DAM 2014) For ny integer $s \geq 1$, if G is a $(2s+1)$-regular graph, then $G \in M_{2s+1}$ iff G is bipartite.

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- Problem For any integer $s \geq 1$, there exists a smallest integer $g(s)$ such that every $g(s)$-edge-disjoint spanning trees is a contraction of a $(2s+1)$-regular bipartite graph.

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- If for any function $b : V(G) \to \mathbb{Z}_{2s+1}$ satisfying $\sum_{v \in V(G)} b(v) \equiv 0$ (mod $2s+1$), G always has an orientation D such that for every vertex $v \in V(G)$, $d_D^+(v) - d_D^-(v) \equiv b(v)$ mod $2s+1$, then G is strongly $\mathbb{Z}_{2s+1}$-connected.

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- If G is strongly $\mathbb{Z}_{2s+1}$-connected, then G is in M_{2s+1}.
- Example (Y. Liang, J. Liu, J. Meng, Y. Shao and HJL, DAM 2014) A complete graph K_n is strongly $\mathbb{Z}_{2s+1}$-connected iff $n \geq 4s+1$.

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- (i) every $k(s)$-edge-connected graph is strongly $\mathbb{Z}_{2s+1}$-connected.
- (ii) every $k(s)$-edge-connected bipartite graph is strongly $\mathbb{Z}_{2s+1}$-connected.

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- Theorem (Lovasz, Thomassen, Wu, Zhang, JCTB 2013, and Wu 2012 Dissertation) If $\kappa'(G) \geq 8s$, then G is strongly $\mathbb{Z}_{2s+1}$-connected.

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- (ii) Conjecture: (SIDAM 2007) Every $(4p+1)$-edge-connected graph is strong $\mathbb{Z}_{2p+1}$-connected.

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- The problem is to show the existence of $R(k, \ell)$ and determine the value of $R(k, \ell)$.

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- (i) G has property $\mathcal{P}$.
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- (iii) G belongs to a well characterized family of graphs (determined by $\mathcal{P}$).

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Recent Progresses: Extremal problems/degree conditions

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- Theorem (R. Luo, R. Xu, J. Yin and G. Yu, EJC 2008) Let G be a simple graph on $n \geq 3$ vertices such that for every pair of nonadjacent vertices u and v in G, $d_G(u) + d_G(v) \geq n$. Then with twelve exceptional graphs, G is strongly $\mathbb{Z}_3$-connected.

Recent Progresses: Extremal problems/degree conditions

- **Theorem** (G. Fan and C. Zhou, DM 2008) Let G be a simple graph on $n \geq 3$ vertices such that $d_G(u) + d_G(v) \geq n$, for every pair of adjacent vertices u and v in G. Then G has a nowhere-zero 3-flow if and only if G is not isomorphic to a $K^+_{3,n-3}$ or to one of the 5 other exceptional graphs.

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- $K^+_{t,n-t}$ is obtained from the complete bipartite graph $K_{t,n-t}$ by adding one new edge joining two independent vertices of degree $n - t$ ($n \geq 2t$).

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- Theorem (X. Zhang et al., DM 2010) Let G be a simple graph on $n \geq 3$ vertices such that $d_G(u) + d_G(v) \geq n$, for every pair of adjacent vertices u and v in G. $G \in M^o_3$ if and only if G is not isomorphic to a member of $\{K_{2,n-2}, K^+_{2,n-2}, K_{3,n-3}, K^+_{3,n-3}\}$ or to one of the 15 other exceptional graphs.

Recent Progresses: Extremal problems/degree conditions

- Theorem (X. Li et al., DM 2012) Let G be a simple 2-edge-connected graph on $n \geq 3$ vertices. If for every $uv \notin E(G)$, $\max\{d_G(u), d_G(v)\} \geq n/2$, then $G \in M_3^o$ if and only if G is not contractible to one of 22 exceptional graphs.

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- Theorem (J. Yan, EJC 2013) Let G be a 2-edge-connected graph of order n such that each pair of vertices x and y at distance 2 satisfies $\max\{d_G(x), d_G(y)\} > \frac{n}{2}$, then either G is strongly $\mathbb{Z}_3$-connected or, with only one exception, G belongs to a family of non strongly $\mathbb{Z}_3$-connected graphs related to the "odd-wheel and fan" family defined in [J. Combin. Theory Ser. B 98 (2008) 1325-1336].

Recent Progresses: Extremal problems/degree conditions

- Observations The theorems above have the following in common (for $s = 1$): under certain degree conditions, either these graphs are strongly $\mathbb{Z}_{2s+1}$-connected, or they can be contracted into a family of finitely many non strongly $\mathbb{Z}_{2s+1}$-connected graphs, or, in some cases, the independence number of the exceptional graphs is unbounded.

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- Next Question: The results above are on strongly $\mathbb{Z}_3$-connected graphs. Can the same be done for strongly $\mathbb{Z}_{2s+1}$-connected graphs with $s \geq 2$?

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- Next Question: The results above are on strongly $\mathbb{Z}_3$-connected graphs. Can the same be done for strongly $\mathbb{Z}_{2s+1}$-connected graphs with $s \geq 2$?
- Next Question: Can the same structural properties be preserved if we replace the lower bounds in the theorems above by an arbitrary non-trivial linear function of n, the number of vertices of the graph?

Recent Progresses: Extremal problems/degree conditions

- Theorem (P.Li and HJL, SIDAM 2014) Let G be a simple graph on n vertices. For any integers $s > 0$ and for any real numbers α and β with $0 < \alpha < 1$, there exist an integer $N = N(\alpha, s)$ and a finite family $\mathcal{F}(\alpha, s)$ of graphs not in M^o_{2s+1} such that if $n \geq N$ and if for every pair of nonadjacent vertices u and v in G, $d_G(u) + d_G(v) \geq \alpha n + \beta$ then either G is strongly $\mathbb{Z}_{2s+1}$-connected or G can be contracted to a member in $\mathcal{F}(\alpha, s)$.

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- Problem This is the first attempt in this direction, and is an Ore-Type condition. How about other degree conditions?

Recent Progresses: Extremal problems/degree conditions

- Observation. If for any $uv \notin E(G)$, $d_G(u) + d_G(v) \geq f(n)$ then $\max\{d_G(u), d_G(v)\} \geq \frac{f(n)}{2}$. This motivates the following study.

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- Theorem (A. Yu, M. Han, J. Liu, HJL, DM 2017) For any integer $s > 0$ and real numbers a, b with $0 < a < 1$, there exist an integer $N = N(a, b, s)$ and a finite family $\mathcal{J}_0(a, s)$ of non-strongly $\mathbb{Z}_{2s+1}$-connected graphs such that for any connected simple graph G with order $n \geq N$, if

$$\text{for any } uv \notin E(G), \ \max\{d_G(u), d_G(v)\} \geq an + b,$$

then G is strongly $\mathbb{Z}_{2s+1}$-connected if and only if G cannot be contracted to a member in $\mathcal{J}_0(a, s)$.

Recent Progresses: Extremal problems/degree conditions

- For any integer n and s, define $f_1(n,s) = \max\{|E(G)| : G$ is simple, $|V(G)| = n$ and $G \notin M^o_{2s+1}$ but for any $e \in E(G)$, $G/e \in M^o_{2s+1}\}$ and

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- $f_2(n,s) = \max\{|E(G)| : G$ is simple, $|V(G)| = n$ and $G \notin M^o_{2s+1}$ but for any $e \in E(G^c)$, $G + e \in M^o_{2s+1}\}$

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- Problem Determine $f_1(n,s)$ and $f_2(n,s)$.

Thank You