Rainbow Ramsey Theorem and Logic

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July 2018

Ramsey's Theorem

Let X be a set. $[X]^n$ is the set of n-element subsets of X. We identify a natural number n with $\{0, 1, \dots, n-1\}$. A k-coloring of $[X]^n$ is a function $f : [X]^n \to k$, and an f-homogeneous set is a set H s.t. f is constant on $[H]^n$.

Theorem (Ramsey)

For each k-coloring f of $[\mathbb{N}]^n$ there is an infinite f-homogeneous H.

For fixed n and k, let RT^n_k denote the corresponding instance of Ramsey's Theorem.

Rainbow Ramsey Theorem

A *k-bounded* coloring of $[X]^n$ is a function $f : [X]^n \to \mathbb{N}$ s.t. every y has at most k preimages. An *f-rainbow* for a k-bounded coloring f is a set R s.t. f is injective on $[R]^n$.

RRT^n_k: every k-bounded coloring of $[\mathbb{N}]^n$ has an infinite rainbow.

Theorem (Galvin)

RT^n_k *implies* RRT^n_k.

Fix a bijection $h : [\mathbb{N}]^n \to \mathbb{N}$. Define $\vec{x} <_o \vec{y}$ iff $h(\vec{x}) < h(\vec{y})$. Given a k-bounded $f : [\mathbb{N}]^n \to \mathbb{N}$, let $g(\vec{x})$ to be the size of $\{\vec{y} <_o \vec{x} : g(\vec{y}) = g(\vec{x})\}$. Then every g-homogeneous set is an f-rainbow.

Question

Does RRT^n_k *imply* RT^n_k*?*

A Context

Most mathematics can be formulated in ZFC, the most popular axiom system of set theory. However, ZFC is too strong as a context to compare theorems, since most reasonable theorems can be proven in ZFC.

Many mathematical notions have important examples with certain countable features, e.g., topological spaces with countable bases, separable metric spaces, continuous functions between topological spaces with countable bases, finite/countable dimensional vector spaces over a countable field, algebraic extensions of countable fields, etc.

Such examples and related propositions can be formulated in the so-called Second Order Arithmetic.

Models of Second Order Arithmetic

In SOA, the mathematical objects under investigation live in models of SOA. A model (of SOA) is a pair $\mathcal{M} = (M, \mathcal{S})$, where M is a model of elementary number theory (i.e., elements of M are 'natural numbers' from an axiomatic viewpoint) and $\mathcal{S}$ is a subset of $\mathcal{P}(M)$ (the powerset of M).

M is the first order part, and $\mathcal{S}$ the second order part.

In many works, $M = \mathbb{N}$.

A Base Axiom System of SOA

The Recursive Comprehension Axiom (RCA_0) usually serves as a base axiom system for analyses of strength of theorems. RCA_0 consists of the following assertions of models $(M, \mathcal{S})$:

- M is the non-negative part of a discrete ordered commutative ring (e.g., $\mathbb{N}$ is the non-negative part of $\mathbb{Z}$);
- mathematical induction holds for subsets of M which are Σ^0_1-definable with parameters from $\mathcal{S}$;
- if $X \in \mathcal{S}$ and Y is computable in X then $Y \in \mathcal{S}$.

Let Φ, Ψ be two formulations of theorems in SOA.

- Φ is stronger than Ψ over RCA_0 ($\mathsf{RCA}_0 + \Phi \vdash \Psi$) iff every model satisfying $\mathsf{RCA}_0 + \Phi$ also satisfies Ψ;
- Φ and Ψ are independent over RCA_0 iff there exist two models $\mathcal{M}_1$ and $\mathcal{M}_2$ s.t. $\mathcal{M}_1$ satisfies $\mathsf{RCA}_0 + \Phi$ but not Ψ and $\mathcal{M}_2$ satisfies $\mathsf{RCA}_0 + \Psi$ but not Φ.

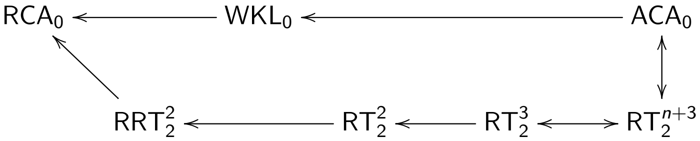
Other Axiom Systems of SOA

ACA_0 is RCA_0 and that $\mathcal{S}$ contains every $Y \subset M$ which is first order definable with parameters from $\mathcal{S}$ (Arithmetic Comprehension).

Between RCA_0 and ACA_0, there is a popular system called WKL_0. Beyond ACA_0, there are two other popular systems ATR_0 and $\Pi^1_1 - \mathsf{CA}_0$.

$\mathsf{RCA}_0, \mathsf{WKL}_0, \mathsf{ACA}_0, \mathsf{ATR}_0, \Pi^1_1 - \mathsf{CA}_0$ are called the Big Five in reverse mathematics.

Ramsey's Theorem in SOA



Jockusch proved: $\mathsf{RT}^{n+3}_2 \leftrightarrow \mathsf{RT}^3_2 \leftrightarrow \mathsf{ACA}_0$, $\mathsf{WKL}_0 \not\to \mathsf{RT}^2_2$.

Seetapun proved: $\mathsf{RT}^2_2 \not\to \mathsf{ACA}_0$.

Lu Liu proved: $\mathsf{RT}^2_2 \not\to \mathsf{WKL}_0$.

RT^2_2 and RRT^2_2

Computablity and Relative Computability

A function $f : \mathbb{N} \to \mathbb{N}$ is computable iff there exists a program F s.t. $f(n) = F(n)$.

The characteristic function of a set X ($\subseteq \mathbb{N}$) is the function

$$\chi_X(n) = \begin{cases} 1 & n \in X; \\ 0 & n \notin X. \end{cases}$$

A set X is computable iff its characteristic function is computable.

Examples: $\mathbb{N}, \emptyset, 2\mathbb{N}$, the set of primes, the set of zeros of a Diophantine equation are computable.

Augment our program language with a special function $M(X, n)$ which gives the characteristic function of the set parameter X (oracle). In the resulting language, we can write program accepting set parameter. If $F(X, n)$ is a program in this language and for a fixed X it gives us a characteristic function of some $Y \subseteq \mathbb{N}$ then Y is computable in X or X-computable (write $Y \leq_T X$, T for Turing).

RT^2_2 and RRT^2_2

The Probability of Computing Infinite Homogeneous Sets/Rainbows

A set can be identified with its characteristic function, which in turns can be identified with a real number in $[0,1]$. So the natural probability measure of $[0,1]$ can be used for $\mathcal{P}(\mathbb{N})$.

Proposition (Mileti)

There exists a computable $f : [\mathbb{N}]^2 \to 2$ s.t. the probability of an oracle in $\mathcal{P}(\mathbb{N})$ computing an infinite f-homogeneous set is 0.

Theorem (Csima and Mileti)

1. *For each k-bounded coloring f of $[\mathbb{N}]^2$, the probability of an oracle computing an infinite f-rainbow is 1.*
2. $\mathsf{RRT}^2_2 \not\to \mathsf{RT}^2_2$.

RT_2^n and RRT_2^n

For a fixed language we can list all programs as $P_0, P_1, \ldots$. Given a set X, we can also list programs with fixed oracle X similarly and denote the n-th program by P_n^X. Write $P_n(x)\downarrow$ if the n-th program halts when taking parameter x.

The halting problem is the set $K = \{n : P_n(n)\downarrow\}$. Similarly define the halting problem relative to X and denote it by K^X. It's well-known that K is not computable and $K^X \not\leq_T X$.

$(\mathbb{N}, \mathcal{S})$ models ACA_0 iff $\mathcal{S}$ contains K^X for every $X \in \mathcal{S}$.

Theorem (WW)

1. *If f is a computable 2-bounded coloring on $[\mathbb{N}]^n$ then there exists an infinite f-rainbow R s.t. $K \not\leq_T R$.*
2. $\mathrm{RRT}_2^n \not\to \mathrm{ACA}_0$, *thus* $\mathrm{RRT}_2^n \not\to \mathrm{RT}_2^3$.

Two New Propositions

From the proof of Csima and Mileti of $\mathrm{RRT}^2_2 \not\to \mathrm{RT}^2_2$, two propositions arise.

The first is so-called $2-\mathrm{WWKL}_0$, which is equivalent to a version of Dominated Convergence Theorem (DCT) that if $(f_n)_n$ is a family of $L^1([0,1])$ functions dominated by g and pointwise converge to $f \in L^1([0,1])$ then $\int f_n \to \int f (n \to \infty)$.

Another is so-called $2-\mathrm{RAN}$ which asserts the existence of a K-random infinite sequence.

Theorem (CM; Conidis and Slaman; Avigad, Dean and Rute)

$2-\mathrm{WWKL}_0 \leftrightarrow \mathrm{DCT} \to 2-\mathrm{RAN} \to \mathrm{RRT}^2_2$.

However, by a theorem of Kučera, models like $(\mathbb{N}, \mathcal{S})$ cannot separate $2-\mathrm{WWKL}_0$ and $2-\mathrm{RAN}$.

Non-standard Models of SOA

A non-standard model of first order arithmetic is the non-negative part of a discrete ordered commutative ring, and looks like $\mathbb{N} + L \times \mathbb{Z}$, where L is a dense linear order without endpoints (e.g., $\mathbb{Q}$). Full mathematical induction fails, but fragments survive.

A set is Σ_1 iff it is of the form $\{x : P_n(x) \downarrow\}$ for some fixed program P_n iff it is the projection (to the first coordinates) of zeros of a Diophantine equation (in several unknowns). The complement of a Σ_1 set is a Π_1 set. In general, the projection of a Π_n set is a Σ_{n+1} set and the complement of a Σ_{n+1} set is Π_{n+1}.

$I\Sigma_n$: mathematical induction holds for Σ_n sets. $I\Sigma_n$ is equivalent to $I\Pi_n$ and part of RCA_0.

$B\Sigma_n$: every Σ_n function (i.e., a function with its graph being a Σ_n set) defined on a finite domain has a finite range.

Theorem (Kirby and Paris)

$I\Sigma_{n+1} \to B\Sigma_{n+1} \to I\Sigma_n$ and the arrows are irreversible.

Weak Pigeonhole Principle

$\Sigma_n - \mathrm{WPHP}$: there exists no Σ_n injection from any $2x$ to x.

For $n > 1$, $B\Sigma_n \to \Sigma_n - \mathrm{WPHP}$.

Roughly, the first order theory of a proposition P is the elementary properties of natural numbers that can be proven from P.

Theorem

1. *(Cholak, Jockusch and Slaman)* $\mathrm{RT}^2_2 \to B\Sigma_2$*;*
2. *(Avigad, Dean and Rute; Conidis and Slaman) The first order theory of* DCT *equals to that of* $B\Sigma_2$*;*
3. *(Belanger, Chong, W., Wong, Yang)* $\Sigma_n - \mathrm{WPHP} \not\to B\Sigma_n$ *when* $n > 1$ *and the first order theory of* $2 - \mathrm{RAN}$ *is bounded by* $I\Sigma_1 + \Sigma_2 - \mathrm{WPHP}$*. Thus* $2 - \mathrm{RAN} \not\to \mathrm{DCT}$*.*

Thanks!