The maximum spectral radius of k-uniform hypergraphs with r pendent vertices

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Let G be a hypergraph with vertex set $V(G)$ and edge set $E(G)$, where $E(G)$ is a set whose elements are subsets of $V(G)$.

If each edge of G contains exactly k distinct vertices, then G is called a k-uniform hypergraph.

An alternating sequence of vertices and edges is called a path if the vertices and edges are distinct, and a cycle if the first and last vertices are the same, the other vertices and all edges are distinct.

If there exists a path between any two vertices of G, then G is called connected.

The adjacency tensor of a k-uniform hypergraph G on n vertices, denoted by $\mathcal{A}(G) = (a_{i_1 \ldots i_k})$, is an order k dimension n symmetric tensor, where

$$a_{i_1 \ldots i_k} = \begin{cases} \frac{1}{(k-1)!}, & \text{if } \{i_1, \ldots, i_k\} \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

For a complex λ, if there exists a nonzero vector $x = (x_1, \ldots, x_n)^T$ such that $\mathcal{A}(G)x = \lambda x^{k-1}$, where $x^{k-1} = (x_1^{k-1}, \ldots, x_n^{k-1})^T$ and $\mathcal{A}(G)x$ is a n-dimensional vector whose i-th component is $(\mathcal{A}(G)x)_i = \sum_{i_2,\ldots,i_k=1}^{n} a_{ii_2\ldots i_k} x_{i_2} \ldots x_{i_k}$, then we call λ an eigenvalue of G.

If λ and x are real, we call λ an H-eigenvalue of G.

Moreover, we call λ an H^+-eigenvalue of G if $x \in R_+^n$ and H^{++}-eigenvalue of G if $x \in R_{++}^n$, where R_+^n is the set of nonnegative vectors of dimension n, and R_{++}^n is the set of positive vectors of dimension n.

The spectral radius of $\mathcal{A}(G)$ is defined as

$$\rho(\mathcal{A}(G)) = \max\{|\lambda| \mid \lambda \in spec(\mathcal{A}(G))\},$$

where $spec(\mathcal{A}(G))$ is the set of all eigenvalues of $A(G)$.

A tensor $\mathcal{T}$ of order k and dimension n is called reducible, if there exists a nonempty proper index subset $I \subset [n]$ such that $T_{i_1 i_2 \dots i_k} = 0$ for any $i_1 \in I$ and any $i_2, \dots, i_k \notin I$.

If $\mathcal{T}$ is not reducible, then $\mathcal{T}$ is called irreducible.

Chang,Pearson and Zhang [3] proved that if $\mathcal{T}$ is irreducible, then $\rho(\mathcal{T})$ is the unique H^{++}-eigenvalue of $\mathcal{T}$, with the unique eigenvector $x \in R_{++}$, up to a positive scaling coefficient.

Yang and Yang [18] proved that if $\mathcal{T}$ is a nonnegative tensor of order k and dimension n, then $\rho(\mathcal{T})$ is an H^+-eigenvalue of $\mathcal{T}$.

A tensor $\mathcal{T}$ of order k and dimension n is called weakly reducible, if there exists a nonempty proper index subset $I \subset [n]$ such that $T_{i_1 i_2 \ldots i_k} = 0$ for any $i_1 \in I$ and at least one of the $i_2, \ldots, i_k \notin I$.

If $\mathcal{T}$ is not weakly reducible, then $\mathcal{T}$ is called weakly irreducible.

Friedland, Gaubert and Han [6] proved that if $\mathcal{T}$ weakly irreducible, then $\rho(\mathcal{T})$ is the unique H^{++}-eigenvalue of $\mathcal{T}$, with the unique eigenvector $x \in R^n_{++}$, up to a positive scaling coefficient.

It is proved that a k-uniform hypergraph G is connected if and only if its adjacency tensor $\mathcal{A}(G)$ is weakly irreducible [6]. Thus for a k-uniform connected hypergraph G, its adjacency tensor $\mathcal{A}(G)$ has a unique positive eigenvector x, called the principal eigenvector of G, with $\|x\|_k = 1$, corresponding to $\rho(G)$.

Lemma

[14] Let $\mathcal{T}$ be a symmetric nonnegative tensor of order k and dimension n. Then

$$\rho(\mathcal{T}) = \max_{\|x\|_k=1} \{x^T(\mathcal{T}x) | x \text{ is a nonnegative n-dimension vector}\}$$

Furthermore, x is an eigenvector of $\mathcal{T}$ corresponding to $\rho(\mathcal{T})$ if and only if it is an optimal solution of the maximization problem of above equation.

By Lemma 1 we know that for a k-uniform connected hypergraph, there is a unique positive vector with $\|x\|_k = 1$ such that $\mathcal{A}(G)x = \rho(G)x^{k-1}$ and

$$\begin{aligned} \rho(G) &= x^T(\mathcal{A}(G)x) = \sum_{i_1,i_2,\ldots,i_k=1}^{n} a_{i_1 i_2 \ldots i_k} x_{i_1} x_{i_2} \ldots x_{i_k} \\ &= k \sum_{e=\{i_1,i_2,\ldots,i_k\}\in E} x_{i_1} x_{i_2} \ldots x_{i_k} \\ &= k \sum_{e\in E(G)} \omega_x(e), \end{aligned} \qquad (1)$$

where $\omega_x(e) = x_{i_1} x_{i_2} \ldots x_{i_k}$ for $e = \{i_1, i_2, \ldots, i_k\}$.

By Equation 1 we have

Lemma

[4, 11]Let G be a connected k-uniform hypergraph and H is a sub-hypergraph of G. Then $\rho(H) \leq \rho(G)$ with equality if and only if $H \cong G$.

Lemma

Let $l \geq 1$, x be the principal eigenvector of a connected k-uniform hypergraph G.

Let G' be obtained from G by deleting $e_1, e_2, \ldots, e_l$ and adding $e_1', e_2', \ldots, e_l'$, where $e_i \in E(G), V_i \subset e_i, U \not\subset e_i, e_i' = \{e_i \backslash V_i\} \cup U$ and $|V_i| = |U|$ for $i = 1, \ldots, l$.

If G' has no multiple edges and $\Pi_{u \in U} x_u \geq \max\limits_{1 \leq i \leq l} \Pi_{v \in V_i} x_v$, then $\rho(G') > \rho(G)$.

Proof.

By Lemma 1 we have

$$\begin{aligned}
\rho(G') &\geq x^T \mathcal{A}(G')x = k \sum_{e \in E(G')} \omega_x(e) \\
&= k \left(\sum_{e \in E(G)} \omega_x(e) + \sum_{i=1}^{l} \left(\omega_x(e_i') - \omega_x(e_i)\right) \right) \\
&= k \left(\sum_{e \in E(G)} \omega_x(e) + \sum_{i=1}^{l} \frac{\omega_x(e_i')}{\prod_{u \in U} x_u} \left(\prod_{u \in U} x_u - \prod_{v \in V_i} x_v\right) \right) \\
&\geq k \sum_{e \in E(G)} \omega_x(e) \\
&= \rho(G).
\end{aligned}$$

□

Lemma

[13]Let x be the principal eigenvector of a connected k-uniform hypergraph G, $v_i, v_j \in V(G)$. If $v_i \in e$ implies $v_j \in e$ for $e \in E(G)$, then $x_{v_j} \geq x_{v_i}$. Furthermore, if there exists an edge which contains v_j but not v_i, then $x_{v_j} > x_{v_i}$.

The complete k-uniform hypergraph on $n \geq k \geq 2$ vertices, denoted by K_n^k, is a hypergraph which has all k-subsets of V as edges. Let $n - r \geq k + 1$, $A_{n,r}$ be the k-uniform hypergraph obtained from the complete hypergraph K_{n-r}^k by adding r pendent vertices and r edges, each new edge contains exactly a new pendent vertex and the same $k - 1$ distinct vertices of $V(K_{n-r}^k)$. Clearly, $A_{n,0} \cong K_n^k$.

Theorem

Let G be a connected k-uniform hypergraph on n vertices with exactly r pendent vertices. If $n - r \geq k + 1$, then $\rho(G) \leq \rho(A_{n,r})$ with equality if and only if $G \cong A_{n,r}$.

Proof: Let G be a k-uniform hypergraph with maximum spectral radius among all connected k-uniform hypergraph on n vertices with exactly r pendent vertices.

Let V^* be the set of pendent vertices in G. Then by Lemma 2, it is easy to obtain that $G[V(G) - V^*]$ is a complete hypergraph.

Let $E^* = \{e_1, e_2, \ldots, e_s\}$ be the set of edges in G, each of which contains at least a pendent vertex, $V_i = e_i \cap V^*$ be the pendent vertex set of e_i, then $s \leq r$ and $V^* = V_1 \cup V_2 \cup \cdots \cup V_s$.

Suppose that $1 \leq |V_1| \leq |V_2| \leq \ldots \leq |V_s|$. Let $N(V_i) = e_i \setminus V_i$. Then

Claim 1. $N(V_1) \supseteq N(V_2) \supseteq \ldots \supseteq N(V_s)$.

Let x be the principal eigenvector of G and $i, j \in [s]$. If there exist vertices u and v such that $u \in N(V_i), v \in N(V_j)$ and $u \notin N(V_j), v \notin N(V_i)$ and $x_u \geq x_v$, then by Lemma 3 $\rho(G_1) > \rho(G)$, where G_1 is obtained from $G \setminus e_j$ by adding the edge $\{e_j \setminus v\} \cup \{u\}$, a contradiction. Thus $N(V_i) \subseteq N(V_j)$ or $N(V_j) \subseteq N(V_i)$ and we complete the proof of Claim 1.

Let $V^0 = V(G) \setminus \{V^* \cup N(V_1)\}$, $N^*(V_i) = N(V_i) \setminus N(V_{i+1})$ for $i = 1, 2, \ldots, s-1$ and $N^*(V_s) = N(V_s)$. Note the fact that $|V^0| + |N(V_1)| = n - r \geq k + 1$ and $|V_1| + |N(V_1)| = k$, then $|V^0| > |V_1|$.

By Lemma 4 we get that $x_u = x_v$ if $\{u, v\} \subseteq V_i (i = 1, \ldots, s)$, V^0 or $N^*(V_j)(j = 1, 2, \ldots, s)$. So we can suppose that $x_w = x_i$ if $w \in V_i (1 \leq i \leq s)$, $x_w = x_0$ if $w \in V^0$ and $x_w = x_i^*$ if $w \in N^*(V_i)(i = 1, 2, \ldots, s)$.

Claim 2. If $N^*(V_i)$ is not empty for some $i \in \{1, 2, \ldots, s-1\}$, then $x_{i+1} > {x_i}^* > x_i$ and $x_s^* > x_s$.

Let $u \in V_i, v \in N^*(V_i)$. Then $x_u = x_i$ and $x_v = x_i^*$. Note that $u, v \in e_i, u \in V^*$ and $v \in V(G) \setminus V^*$, that is, $u \in e_i$ implies $v \in e_i$, and $d(v) \geq 2$, by Lemma 4 we get ${x_i}^* > x_i$.

Now we assume that $x_i^* \geq x_{i+1}$. Since $N^*(V_i)$ is not empty, $|N(V_i) \setminus N(V_{i+1})| > 0$ and $|V_{i+1}| > |V_i|$. Let V_{i+1}^1, V_{i+1}^2 be the two partition of V_{i+1} such that $|V_{i+1}^1| = |V_i|$.

Let G_2 be obtained from G by deleting e_{i+1} and adding the edges $e^*_{i+1} = V^1_{i+1} \cup N(V_i)$ and e^0, where e^0 is the edge containing V^2_{i+1} and any $k - |V^2_{i+1}|$ vertices of $V(G) - V^*$.

Clearly, $\frac{\omega(e^*_{i+1})}{\omega(e_{i+1})} = (\frac{x^*_i}{x_{i+1}})^{|N(V_i)\setminus N(V_{i+1})|} \geq 1$.

G_2 is a connected hypergraph with exactly r pendent vertices, and $\rho(G_2) \geq k\Sigma_{e\in E(G_2)}\omega_x(e) = k\Sigma_{e\in E(G)\setminus e_{i+1}}\omega_x(e) + k\omega(e^*_{i+1}) + k\omega(e^0) \geq k\Sigma_{e\in E(G)}\omega_x(e) + k\omega(e^0) = \rho(G) + k\omega(e^0) > \rho(G)$, a contradiction.

Claim 3. $|V_1| = |V_2| = \ldots = |V_s|$.

Assume that $|V_s| > |V_1|$. By Claims 1 and 2 we can get that $x_s > x_1$ and $x_s > x_i^*$ for any $i < s$ with $|N^*(V_i)| \neq 0$. By Claim 1 we know that $N(V_1) \supseteq N(V_s)$ and $|e_1 \backslash N^*(V_s)| = k - |N^*(V_s)| = |V_s|$. Then there is a bijective σ from $e_1 \backslash N^*(V_s)$ to V_s. Let x^* be obtained from x by exchanging the weight of $e_1 \setminus N^*(V_s)$ and V_s, that is, $x_u^* = x_{\sigma(u)}, x_v^* = x_{\sigma^{-1}(v)}$ for $u \in e_1 \setminus N^*(V_s), v \in V_s$, and $x_w^* = x_w$ for $w \in V(G) \setminus \{\{e_1 \setminus N^*(V_s)\} \cup V_s\}$. Clearly, $\|x^*\|_k = \|x\|_k = 1$, $\omega_{x^*}(e_1) + \omega_{x^*}(e_s) = \omega_x(e_1) + \omega_x(e_s)$, and $\omega_{x^*}(e) \geq \omega_x(e)$ for any $e \in E(G) \setminus \{e_1, e_s\}$.

Let

$$E^0(G) = \{e | e \in E(G), e \subseteq V(G) \setminus V^*, e \cap \{e_1 \setminus N^*(V_s)\} \neq \emptyset\}.$$

Clearly, $e_1, e_s \notin E^0(G)$. Note that $n - r \geq k + 1$. Then $|E^0(G)| \neq 0$ and $\omega_{x^*}(e) > \omega_x(e)$ for any $e \in E^0(G)$. We get

$$\begin{aligned} \rho(G) &= k \sum_{e \in E(G)} \omega_x(e) \\ &\leq k \sum_{e \in E(G) \setminus E^0(G)} \omega_{x^*}(e) + k \sum_{e \in E^0(G)} \omega_x(e) \\ &< k \sum_{e \in E(G) \setminus E^0(G)} \omega_{x^*}(e) + k \sum_{e \in E^0(G)} \omega_{x^*}(e) \\ &= k \sum_{e \in E(G)} \omega_{x^*}(e) \\ &\leq \rho(G), \end{aligned}$$

a contradiction.

By Claim 3 we can get that $N(V_1) = N(V_2) = \ldots = N(V_s)$ and $x_u = x_s^*$ for any $u \in N(V_1)$. It is easy to prove that $x_1 = \ldots = x_s$. By Claim 2 we get that $x_1 < x_s^*$. Note that $n - r \geq k + 1$ and $|V^0| > |V_1|$. For any $v \in V^0$,

$$\begin{aligned} \rho(G)x_1^{k-1} &= x_1^{|V_1|-1}(x_s^*)^{k-|V_1|}, \\ \rho(G)x_0^{k-1} &= \sum_{v\in e} \frac{\omega_x(e)}{x_0} > x_0^{|V_1|-1}(x_s^*)^{k-|V_1|}, \end{aligned}$$

and $x_0 > x_1$.

If $|V_1| \geq 2$, let G_3 be obtained from G by deleting e_1 and adding $|V_1|$ edges, each contains exactly a vertex of V_1, any $|V_1|-1$ vertices of V^0 and $N(V_1)$. It is easy to see that

$$\begin{aligned}\rho(G_3) &\geq k\sum_{e\in E(G_3)}\omega_x(e)\\ &> k\sum_{e\in E(G)}\omega_x(e)+(|V_1|-1)x_1x_0^{|V_1|-1}(x_s^*)^{k-|V_1|}\\ &> \rho(G),\end{aligned}$$

a contradiction. Thus $|V_1| = \ldots = |V_s| = 1$ and $s = r$. By Claims 1 and 3 we get that $G \cong A_{n,r}$.

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Thank You!