Ricci-flat Graphs with Girth Four

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A Little Theoretical Physics and Geometry

- According to the superstring theory, we live in a 10-dimensional world: a 4-dimensional Einstein's spacetime with an extra 6-dimensional Calabi-Yau(丘成桐) manifold.
- One main feature of Calabi-Yau manifold is that it is Ricci-flat everywhere.

Extending the notion of Ricci Curvature

J. Lott and C. Villani extend the notion of Ricci Curvature from the setting of smooth Riemannian manifold to the setting of metric-measure spaces (Ann. of Math., 2009).

Different Concepts of Curvature on Graphs

- F. Chung(金芳蓉) and S.-T. Yau (1996)
- Y. Ollivier (2009)
- Y. Lin(林勇), L. Lu(陆临渊) and S.-T. Yau (2011)

Distribution $m_x^\alpha(v)$

Given a simple graph $G = (V, E)$, a probability distribution is a function $m : V \to [0, 1]$ with $\sum_{x \in V} m(x) = 1$. To define Ricci curvature for each edge of the graph, we only consider distributions m_x^α in the following form,

$$m_x^\alpha(v) = \begin{cases} \alpha, & v = x; \\ \frac{1-\alpha}{d_x}, & v \in N(x); \\ 0, & \text{otherwise}, \end{cases}$$

where $\alpha \in [0, 1]$ and $x \in V$.

Transportation Problem

Let $xy \in E$, and let m_x^α and m_y^α be two distributions. A transportation problem between the two distributions can be stated as a linear programming problem. That is, to find the minimum transportation distance

$$\min \sum_{u,v \in V} d(u,v) X_{uv},$$

subject to the constraints

$$\begin{cases} \sum_{v \in V} X_{uv} = m_x^\alpha(u), & u \in V; \\ \sum_{u \in V} X_{uv} = m_y^\alpha(v), & v \in V; \\ X_{uv} \geqslant 0, \end{cases}$$

where the variable X_{uv} denotes the amount transfered from vertex u to vertex v.

Optimal Solution to the Transportation Problem

Define the transportation distance between m_x^α and m_y^α to be optimal solution to the above linear programming problem, namely

$$W(m_x^\alpha, m_y^\alpha) = \min \sum_{u,v \in V} d(u,v) X_{uv}.$$

Ricci Curvature

For any edge $xy \in E$, the Ricci curvature $\kappa(x, y)$ is defined to be

$$\kappa(x, y) = \lim_{\alpha \to 1} \frac{1 - W(m_x^\alpha, m_y^\alpha)}{1 - \alpha}.$$

A graph G is Ricci-flat if $\kappa(x, y) = 0$ for all edges $xy \in E$.

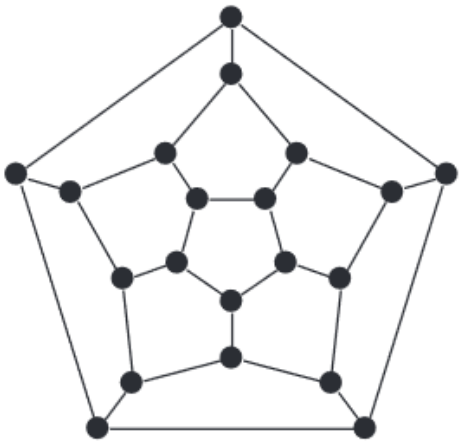
Ricci Curvature - some examples

- For complete graphs K_n, $\kappa(x, y) = \frac{n}{n-1}$ for each edge xy.
- For cycle C_4, $\kappa(x, y) = 1$ for each edge xy.
- For cycle C_5, $\kappa(x, y) = 0.5$ for each edge xy.
- For cycle $C_n (n \geq 6)$, $\kappa(x, y) = 0$ for each edge xy.
- Ricci curvature can be negative.

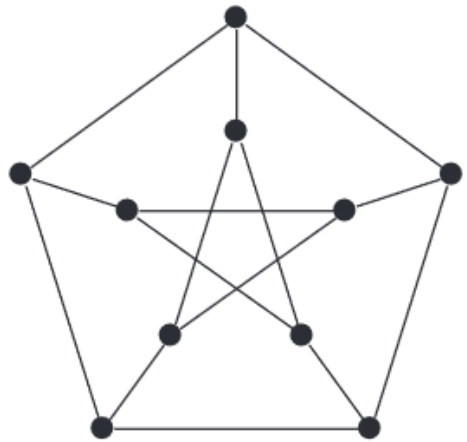
Ricci-flat Graphs with Girth at least 5

Theorem (Lin-Lu-Yau, 2014)

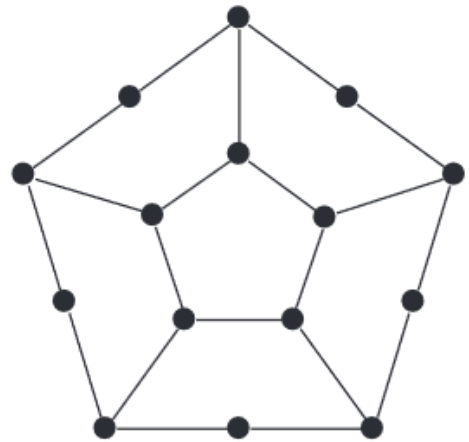
A Ricci-flat graph with girth at least five is isomorphic to: (1) the infinite path, (2) a cycle of length at least six, (3) the dodecahedral graph, (4) the half-dodecahedral graph, or (5) the Petersen graph.



Dodecahedral graph



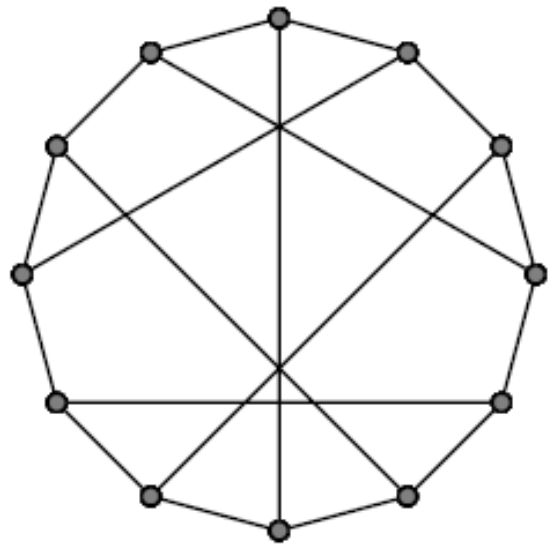
Petersen graph



Half-dodecaheral graph

Ricci-flat Graphs with Girth at least 5: Erratum

In 2018, Cushing, Kangaslampi, and Shiping Liu(刘世平) find Lin-Lu-Yau missed one Ricci-flat graph of girth 5, the Triplex graph. They write a Erratum together.



Triplex

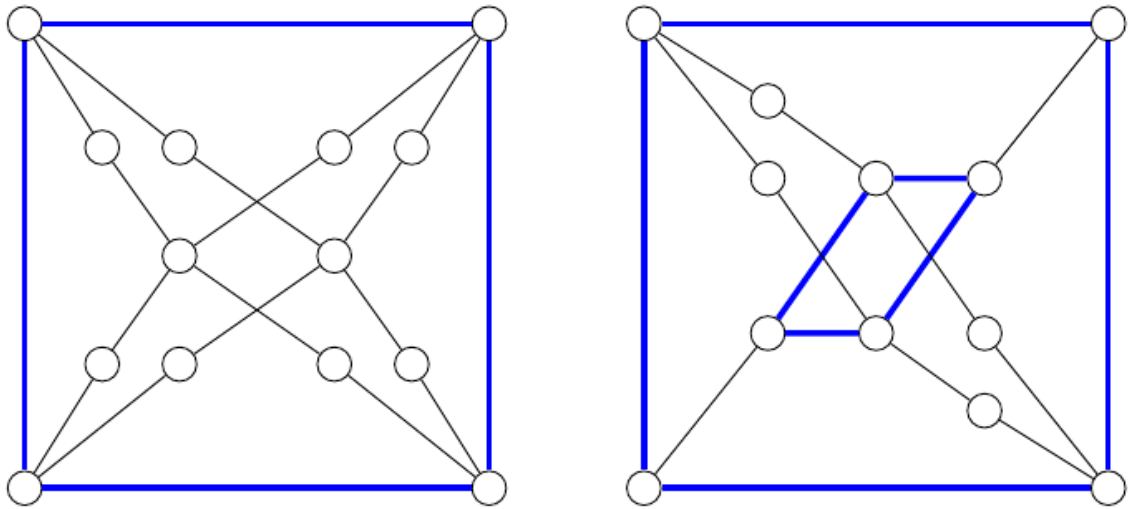
Ricci-flat graphs with Girth 4 and 3

There are infinitely many Ricci-flat graphs with girth 4 or 3.

Our Results

Theorem (He-Luo-Y.-Yuan, 2017)

A Ricci-flat graph with girth four such that no vertex is shared by two 4-cycles is isomorphic to one of the following two graphs.



Thank you!