The pessimistic diagnosability of graphs and its applications to four kinds of interconnection networks

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Shaoguan, July 5-7, 2019

Background

In multiprocessor systems, the systems always contain a large number of processors. Some processors may fail when a system is in put into use. It is important to identify the faulty processors. The process of identifying the faulty processors is called the *diagnosis* of the system.

The maximal number of faulty processors that a system can guarantee to diagnosis is called the *degree of diagnosability* of the system.

Background

The t/t-**diagnosable system** is called ***the pessimistic diagnosis strategy***, which is proposed by Kavianpour and Friedman.

In the pessimistic diagnosis strategy, all the faulty vertices can be isolated into a faulty set with ***at most one fault-free vertex may be contained in the faulty set.***

Definitions

- pessimistic diagnosability

  The *pessimistic diagnosability*, denoted by $t_p(G)$, of a systems G, is the maximal number of faulty processors so that the system G is t/t-diagnosable.

theorem A

The following theorem is a sufficient and necessary for a graph G to be t/t-diagnosable.

Theorem (C.-H. Tsai, J.-C. Chen, Theor. Comput. Sci. 501 (2013), pp. 62-71.)

*A graph G is **t/t-diagnosable** if and only if for each vertex set $S \subseteq V(G)$ with $|S| = p$, $0 \leq p \leq t-1$, $G - S$ has at most one trivial component and each nontrivial component C of $G - S$ satisfies $|V(C)| \geq 2(t-p)+1$.*

Definitions

For any two vertices u and v in a graph G, the number of **common neighbors** of u and v is denoted by $C_G(u, v)$, i.e., $C_G(u, v) = |N_G(u) \cap N_G(v)|$.

The ***distance*** between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest path between them.

The ***girth*** of a graph G is the the length of the shortest cycle in G.

The ***components*** of a graph G are the maximally connected subgraphs.

A component is ***trivial*** if it is one vertex, otherwise it is *nontrivial*.

The ***connectivity*** of a graph G, denoted by $\kappa(G)$, is the minimum number of vertices whose removal will result a disconnected or a trivial graph.

Main result

Theorem

In a simple graph $G = (V(G), E(G))$, let n_0 be the minimum cardinality of the neighborhoods of any two adjacent vertices, i.e., $n_0 = min\{|N_G(\{u, v\})| | (u, v) \in E(G)\}$. Let $\kappa(G)$ be the connectivity of G. The pessimistic diagnosability of G is

$$t_p(G) = n_0$$

if the following two conditions hold:
(1) For any subset $U \subset V(G)$ with $2 \leq |U| \leq 2(n_0 - \kappa(G))$, $|N_G(U)| \geq n_0$;
(2) $|V(G)| \geq 2n_0 + \kappa(G)$.

Sketch of proof

We first prove that $t_p(G) \leq n_0$.
The proof is by contradiction. Assume that $t_p(G) \geq n_0 + 1$. Let $e = (u, v)$, $H = \{u, v\}$ and $S = N_G(H)$ such that $|S| = |N_G(H)| = n_0 \leq t_p(G) - 1$. Since H induces an edge which is a nontrivial component of $G - S$, by Theorem A, $|H| \geq 2[t_p(G) - |S|] + 1 = 2[t_p(G) - n_0] + 1 \geq 2 + 1 = 3$. However, $|H| = |\{u, v\}| = 2$. That is a contradiction. Hence, $t_p(G) \leq n_0$.

Sketch of proof

Now we prove that $t_g(G) \geq n_0$.
We consider the following two cases.
Case 1. $G - S$ contains more than one trivial components.
Case 2. $G - S$ contains a nontrivial component C with $|V(C)| \leq 2(n_0 - p)$.
Case 2.1. $0 \leq p \leq \kappa(G) - 1$.
Case 2.2. $\kappa(G) \leq p \leq n_0 - 1$.

lemma A

Lemma (J. Fan, IEEE Trans. Parallel Distrib. Syst. 40 (1) (1991), pp. 88-93.)

Let G be a connected graph and $U \subset V(G)$. Then $|N_{V(G)-U}(U)| \geq \kappa(G)$ if $|V(G) - U| \geq \kappa(G)$, otherwise, $|N_{V(G)-U}| = |V(G) - U|$.

Application to hypercube-like networks

Definition (A. S. Vaidya, P.S.N. Rao, S.R. Shankar, In Proc. 5th IEEE Symp. Parallel Distrib. Process. 1993, PP. 800-803.)

An n-dimensional hypercube-like networks, denoted by XQ_n, is constructed by recursive method.
$XQ_1 = K_1$.
XQ_n is constructed by two copies of XQ_{n-1}, denoted by XQ^0_{n-1} and XQ^1_{n-1}, and by adding some perfect matchings between XQ^0_{n-1} and XQ^1_{n-1}.

Application to hypercube-like networks

Lemma ([6, 26])

(1) $V(XQ_n) = 2^n$. *(2)* $\kappa(XQ_n) = n$.

Lemma ([8])

For $U \subset V(XQ_n)$, *if* $|U| = k$, $1 \leq k \leq n+1$, $n \geq 1$, *then* $|N_{XQ_n}(U)| \geq kn - k(k+1)/2 + 1$.

Lemma

For any edge $e = (u, v)$ *of* XQ_n, $|N_{XQ_n}(\{u, v\})| = 2n - 2$, *where* $n \geq 2$.

Application to hypercube-like networks

Lemma ([29])

For $U \subset V(XQ_n)$ with $3 \leq |U| \leq 2^n - 2n - 1$, then $|N_{XQ_n}(U)| > 2n - 2$, where $n \geq 5$.

Lemma

For $U \subset V(XQ_n)$ with $2 \leq |U| \leq 2n - 2$, then $|N_{XQ_n}(U)| \geq 2n - 2$, where $n \geq 4$.

Lemma

$|V(XQ_n)| \geq 2(2n - 2) + n$, where $n \geq 4$.

Application to hypercube-like networks

Theorem

$t_p(XQ_n) = 2n - 2$, where $n \geq 4$.

Application to dual-cube

Definition ([3])

DC_n consists of 2^{n+1} copies of Q_n with two classes, named Class 0 and Class 1. Each class consists of 2^n copies of Q_n and each copy is called a cluster. Each vertex is labeled by $u_{2n}u_{2n-1}u_{2n-2}\dots u_nu_{n-1}\dots u_0$ with $u_{2n-1}u_{2n-2}\dots u_n$ is cluster id and $u_{n-1}u_{n-2}\dots u_0$ is vertex id. If $u_{2n}=0$, then it is in Class 0; if $u_{2n}=1$, then it is in Class 1. Two vertices $u=u_{2n}u_{2n-1}\dots u_0$ and $v=v_{2n}v_{2n-1}\dots v_0$ are adjacent if and only if the following conditions hold:
(1) u and v differ in exactly one bit position i, where $0\le i\le 2n$;
(2) if $0\le i\le n-1$, then $u_{2n}=v_{2n}=0$;
(3) if $n\le i\le 2n-1$, then $u_{2n}=v_{2n}=1$.

Application to dual-cube

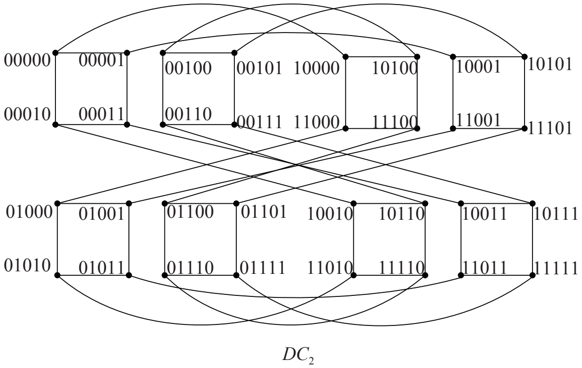

Fig.: The illustration of DC_2.

Application to dual-cube

Lemma ([15, 16, 17])

(1) DC_n *has* 2^{2n+1} *vertices.*
(2) DC_n *is* $(n+1)$*-regular graph.*
(3) $\kappa(DC_n) = n+1$.

Lemma ([28])

For any two distinct vertices u *and* v *in n-dimensional dual-cube* DC_n*, if* $d(u,v)=2$ *then* $C(u,v)\leq 2$*, otherwise if* $d(u,v)=1$ *or* $d(u,v)\geq 3$*, then* $C(u,v)=0$.

Lemma

For any edge $e=(u,v)$ *of* DC_n*,* $|N_{DC_n}(\{u,v\})|=2n$*, where* $n\geq 2$.

Application to dual-cube

Lemma

For any two vertices u and v in DC_n, $|N_{DC_n}(\{u, v\})| \geq 2n$.

Lemma

Let $U \subset V(DC_n)$ *with* $2 \leq |U| \leq 2n - 2$. *Then* $|N_{DC_n}(U)| \geq 2n$, *where* $n \geq 4$.

Lemma

$|V(DC_n)| > 2 \cdot 2n + (n + 1)$, *where* $n \geq 4$.

Application to dual-cube

Theorem

$t_p(DC_n) = 2n$, *where* $n \geq 4$.

Application to pancake graph

Definition ([22])

An n-dimensional pancake graph is denoted by $P_n = (V(P_n), E(P_n))$, where $V(P_n)$ is the set of all permutations of $\langle n \rangle$, where $\langle n \rangle = \{1, 2, \ldots, n\}$, and the edge set $E(P_n) = \{(u, (u)^i) | u = u_1 \ldots u_i \ldots u_n, (u)^i = u_i u_{i-1} \ldots u_2 u_1 u_{i+1} \ldots u_n, 2 \leq i \leq n\}$.

Application to pancake graph

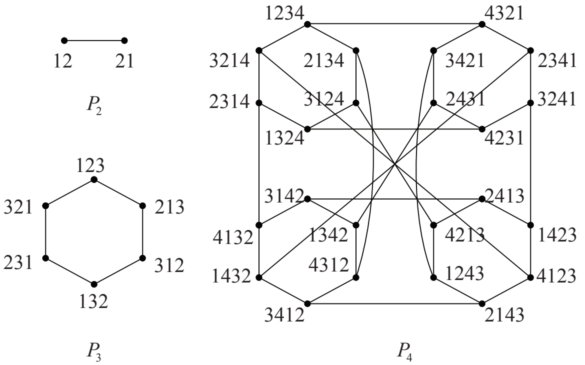

Fig.: Illustrations of P_2, P_3 and P_4.

Application to pancake graph

Lemma ([1, 13, 18, 23])

(1) P_n is $(n-1)$-regular with $n!$ nodes.
(2) $\kappa(P_n) = n - 1$.
(3) The girth of P_n is 6, where $n \geq 3$.
(4) P_n can be decomposed into n vertex-disjoint subgraphs, denoted by P_n^i, by fixing the symbol in the last position n, in which the symbol in the nth position is i, where $i \in \langle n \rangle$. P_n^i is isomorphic to P_{n-1}.

Application to pancake graph

Lemma

Let u and v be any two vertices in P_n with $n \geq 3$. If $d(u,v) = 2$, then $C(u,v) = 1$, otherwise, if $d(u,v) = 1$ or $d(u,v) \geq 3$, then $C(u,v) = 0$.

Lemma

For any edge $e = (u,v)$ of P_n, $|N_{P_n}(\{u,v\})| = 2n-4$, where $n \geq 3$.

Lemma

Let u and v be any two vertices in P_n. Then $|N_{P_n}(\{u,v\})| \geq 2n-4$, where $n \geq 3$.

Application to pancake graph

Lemma

$|V(P_n)| > 2(2n-4) + (n-1)$, *where* $n \geq 4$.

Lemma

For any subset $U \subset V(P_n)$ *with* $2 \leq |U| \leq 2n-6$, *then* $|N_{P_n}(U)| \geq 2n-4$, *where* $n \geq 4$.

Application to pancake graph

Theorem

$t_p(P_n) = 2n - 4$, *where* $n \geq 4$.

Application to burnt pancake network

Definition ([4])

An n-dimensional burnt pancake network is denoted by $BP_n = (V(BP_n), E(BP_n))$, where $V(BP_n)$ is the set of all signed permutations of $\langle n \rangle$, and the edge set is

$E(BP_n) = \{(u, \overline{(u)}^i) | u = u_1 \dots u_i \dots u_n, \overline{(u)}^i = \overline{u_i u_{i-1} \dots u_2 u_1} u_{i+1} \dots u_n, 1 \leq i \leq n\}$.

Application to burnt pancake network

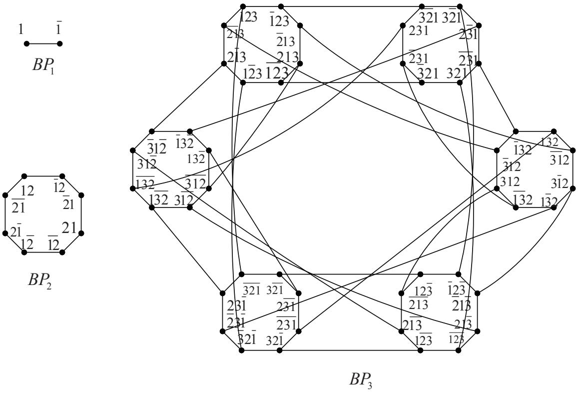

Fig.: Illustrations of BP_1, BP_2 and BP_3.

Application to burnt pancake network

Lemma ([4, 12])

(1) BP_n is n-regular with $n! \times 2^n$ nodes.
(2) $\kappa(BP_n) = n$.
(3) The girth of BP_n is 8, where $n \geq 2$.
(4) BP_n can be decomposed into $2n$ vertex-disjoint subgraphs, denoted by BP_n^i, by fixing the symbol in the last position n, in which the symbol in the nth position is i, where $i \in \langle n \rangle$. BH_n^i is isomorphic to BP_{n-1}.

Application to burnt pancake network

Lemma

For any two vertices u and v in BP_n, if $d(u, v) = 2$, then $C(u, v) = 1$, otherwise if $d(u, v) = 1$ or $d(u, v) \geq 3$, then $C(u, v) = 0$.

Lemma

For any edge $e = (u, v)$ in BP_n, $|N_{BP_n}(\{u, v\})| = 2n - 2$, where $n \geq 2$.

Lemma

For any two vertices u and v in BP_n, $|N_{BP_n}(\{u, v\})| \geq 2n - 2$, where $n \geq 2$.

Application to burnt pancake network

Lemma

Let $U \subset V(BP_n)$ with $2 \leq |U| \leq 2n-4$. Then $|N_{BP_n}(U)| \geq 2n-2$, where $n \geq 3$.

Lemma

$|V(BP_n)| > 2(2n-2)+n$, where $n \geq 3$.

Application to burnt pancake network

Theorem

$t_n(BP_n) = 2n - 2$, where $n \geq 3$.

Concluding remarks

Table: Applications of main theorem

G	$V(G)$	n_0	$\kappa(G)$	$t_p(G)$
XQ_n	2^n	$2n-2$	$2n-2$	$2n-2$
DC_n	2^{2n+1}	$2n$	$n+1$	$2n$
P_n	$n!$	$2n-4$	$n-1$	$2n-4$
BP_n	$n! \times 2^n$	$2n-2$	n	$2n-2$

[1] S.B. Akers, B. Krishnamurthy, *A group-theoretic model for symmetric interconnection networks*, IEEE Trans. Computers. 38 (4) (1989), pp. 555-566.

[2] A. Bondy, U.S.R. Murty, *Graph theory*, Springer, New York, 2007.

[3] S.-Y. Chen, S.-S. Kao, *Hamiltonian connectivity and globally 3*-connectivity of dual-cube extensive networks*, Comput. Electr. Eng. 36 (2010), pp. 404-413.

[4] C. Chin, T.-H. Weng, L.-H. Hsu, S.-C. Chiou, *The spanning connectivity of the burnt pancake graphs*, IEICE Trans. Inform. Syst.E92-D(3) (2009), pp. 389-400.

[5] K.Y. Chwa, S.L. Hakimi, *On fault identification in diagnosable systems*, IEEE Trans. Computers. 30 (6) (1981), pp. 414-422.

[6] J. Fan, L. He, *BC interconnection networks and their properties*, Chinese Journal of Computers-Chinese Edition. 26 (1) (2003), pp. 84-90.

- [7] J. Fan, *Diagnosability of the Möbius cubes*, IEEE Trans. Parallel Distrib. Syst. 40 (1) (1991), pp. 88-93.
- [8] J. Fan, X. Lin, *The t/k-diagnosability of the BC graphs*, IEEE Trans. Comput. 54 (2) (2005), pp. 176-184.
- [9] M.-M. Gu, R.-X. Hao, Y.-Q. Feng, *The pessimistic diagnosability of bubble-sort star graphs and augmented k-ary n-cubes*, Int. J. Comput. Math. 1 (2016), pp. 98-112.
- [10] M.-M. Gu, R.-X. Hao, *The pessimistic diagnosability of three kinds of graphs*, Discrete Appl. Math. 217 (2017), pp. 548-556.
- [11] R.-X. Hao, M.-M. Gu, Y.-Q. Feng, *The pessimistic diagnosabilities of some general regular graphs*, Theor. Comput. Sci. 609 (2016), pp. 413-420.
- [12] T. Iwasaki, K. Kaneko, *Fault-tolerant routing in burnt pancake graphs*, Inform. Process. Lett. 110 (2010), pp. 535-538.
- [13] A. Kanevsky, C. Feng, *On the embedding of cycles in pancake graphs*, Parallel Comput. 1995, pp. 923-936.

- [14] A. Kavianpour, A.D. Friedman, *Efficient design of easily diagnosable systems*, In Proc. 3rd USA-Japan Comput. Conf. 1978, pp. 251-257.
- [15] Y. Li, S. Peng, *Dual-cubes: A new interconnection networks for high-performance computer cluster*, Proceedings of the 2000 International Computer Symposium, Workshop on Computer Architecture. 2000, pp. 51-57.
- [16] Y. Li, S. Peng, W. Chu, *Hamiltonian cycle embedding for fault tolerance in dual-cube*, IASTED International Conference on Networks, Parallel. Distr. Process. Appl. 2002, pp. 1-6.
- [17] Y. Li, S. Peng, W. Chu, *Fault-tolerant cycle embedding in dual-cube with node faults*, Int. J. High Perform. Comput. Networking. 3 (1) 2005, pp. 45-53.
- [18] C.-K. Lin, H.-M. Huang, L.-H. Hsu, *The super connectivity of the pancake graphs and the super laceability of the star graphs*, Theor. Comput. Sci. 339 (2005), pp. 251-271.

- [19] J. Maeng, M. Malek, *A comparison connection assignment for self-diagnosis of multiprocessors systems*, In Proc. 11th Int. Symp. Fault-Tolerant Computing. 1981, pp. 173-175.
- [20] F.P. Preparata, G. Metze, R.T. Chien, *On the connection assignment problem of diagnosis systems*, IEEE Trans. Electron. Comput. 16 (12) (1967), pp. 848-854.
- [21] A. Sengupta, A.T. Dahbura, *On self-diagnosable multiprocesor systems, Diagnosis by the comparison approach*, IEEE Trans. Computers. 41 (11) (1992), pp. 1386-1396.
- [22] S. Song, X. Li, S. Zhou, M. Chen, *Fault tolerance and diagnosability of burnt pancake networks under the comparision model*, Theor. Comput. Sci. 582 (2015), pp. 48-59.
- [23] Y. Suzuki, K. Kaneko, *An algorithm for node-disjoint paths in pancake graphs*, IEICE Trans. Inform. Syst. 3 (2003), pp. 610-615.

[24] C.-H. Tsai, J.-C. Chen, *Fault isolation and identification in general biswapped networks under the PMC diagnostic model*, Theor. Comput. Sci. 501 (2013), pp. 62-71.

[25] C.-H. Tsai, *The pessimistic diagnosability of alternating group graphs under the PMC model*, Inform. Process. Let. 115 (2015), pp. 151-154.

[26] A. S. Vaidya, P.S.N. Rao, S.R. Shankar, *A class of hypercube-like networks*, In Proc. 5th IEEE Symp. Parallel Distrib. Process. 1993, PP. 800-803.

[27] C.L. Yang, G.M. Marson, R. Leonetti, *On fault identification and isolation in t_1/t_1-diagnosable systems*, IEEE Trans. Computers. 35 (7) (1986), pp. 639-644.

[28] X. Yang, S. Zhou, *On conditional fault tolerant of dual-cubes*, Int. J. Parallel. Emergent. Distrib. Systl. 28 (3) (2013), pp. 199-213.

[]29L .-C. Ye, J.-R. Liang, H.-X. Lin, *A fast pessimistic diagnosis algorithm for hypercube-like networks under the comparison model*, IEEE Trans. Comput. 65 (9) (2016), pp. 2884-2888.