Robust Hamiltonicity of line graphs and compatible Euler tours in Eulerian graphs

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Shaoguan, Jul. 6th, 2019

Robust Hamiltonicity

- If a graph G has property $\mathcal{P}$, then how strongly does G possess $\mathcal{P}$?(robustness of the property).
- For Hamiltonicity, two ways to measure the robustness:
  - Different Hamiltonian cycles or edge-disjoint Hamiltonian cycles;
  - Resilience: compute the robustness in terms of the number of edges one must delete from G locally or globally in order to destroy the property P (similar to fault-tolerance).

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Robust Hamiltonicity of Dirac Graphs

Theorem (Cuckler, Kahn, 2009)

Every Dirac graph contains at least $\frac{n!}{(2+o(1))^n}$ Hamiltonian cycles.

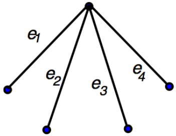
Theorem (Krivelevich, Lee, Sudakov, 2016)

There exists a positive constant C such that for $p \geq \frac{C \log n}{n}$ and a graph G on n vertices of minimum degree at least $\frac{n}{2}$, the random subgraph G_p is a.a.s. Hamiltonian.

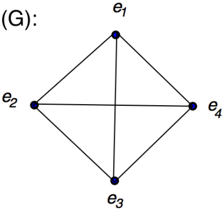
- G_p: the probability space of graphs obtained by taking every edge of G independently with probability p.

Line graph:

G:



L(G):



Theorem (Harary and Nash-Williams, 1965)

Let G be a graph not a star. Then $L(G)$ is Hamiltonian if and only if G has a dominating closed trail.

Hamiltonicity of line graphs

Conjecture (Thomassen, 1986)

Every 4-connected line graph is Hamiltonian.

Conjecture (Matthews and Sumner, 1984)

Every 4-connected claw-free graph is Hamiltonian.

Theorem (Ryjáček, 1997)

Let G be a claw-free graph. Then

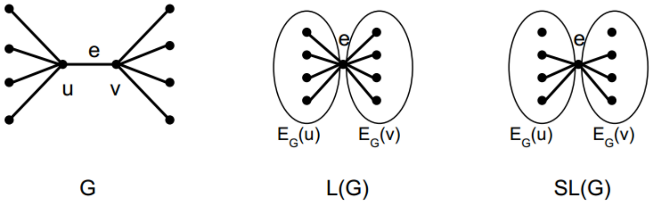
1. *the closure $cl(G)$ is well-defined.*
2. *$cl(G)$ is the line graph of a triangle-free graph.*
3. *$c(G) = c(cl(G))$.*

If $L(G)$ is Hamiltonian, then

- can we remove some edges in $L(G)$ such that the resulting graph is Hamiltonian?
- how many edge-disjoint Hamiltonian cycles in $L(G)$?

DEFINITION OF $SL(G)$

- it's a spanning subgraph of $L(G)$,
- every vertex $e = uv$ is adjacent to at least $min\{d_G(u) - 1, \lceil \frac{3}{4} d_G(u) + \frac{1}{2} \rceil\}$ vertices of $E_G(u)$ and to at least $min\{d_G(v) - 1, \lceil \frac{3}{4} d_G(v) + \frac{1}{2} \rceil\}$ vertices of $E_G(v)$.



- $\mathcal{SL}(G)$ denote this graph family.

Hamiltonian cycles in SL(G)

Theorem (Bai, He, Li, Yang, 2016)

If $L(G)$ is Hamiltonian, then every $SL(G) \in \mathcal{SL}(G)$ is also Hamiltonian.

Hamiltonian cycle decomposition

Hamiltonian cycle decomposition:

- if G is even regular and $E(G)$ is the edge-disjoint union of Hamiltonian cycles;
- if G is odd regualr and $E(G)$ is the edge-disjoint union of Hamiltonian cycles and a 1-factor.

THEOREM (JAEGER, 1983)

If G has a Hamiltonian cycle decomposition into two Hamiltonian cycles, then $L(G)$ has a Hamiltonian cycle decomposition into three Hamiltonian cycles.

BERMOND'S CONJECTURE

CONJECTURE (BERMOND, 1990)

If G has a Hamiltonian cycle decomposition, then $L(G)$ also has a Hamiltonian cycle decomposition.

THEOREM (MUTHUSAMY, PAULRAJA, 1995)

If G has a Hamiltonian decomposition into an even number of Hamiltonian cycles, then $L(G)$ admits a Hamiltonian cycle decomposition.

THEOREM (MUTHUSAMY, PAULRAJA, 1995)

If G has a Hamiltonian decomposition into an odd number of Hamiltonian cycles, then the edge set of $L(G)$ can be partitioned into Hamiltonian cycles and a 2-factor.

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Bermond's conjecture

Theorem (Pike, 1995)

If G is a 5-regular Hamiltonian decomposable graph, then $L(G)$ admits a Hamiltonian cycle decomposition.

Theorem (Verrall, 1998)

$L(K_{2n})$ has a Hamiltonian cycle decomposition.

Theorem (Pike, 1995)

If G is a bipartite $(2k+1)$-regular graph that has a Hamiltonian cycle decomposition, then $L(G)$ admits a Hamiltonian cycle decomposition.

Compatible Euler tours

Compatible Euler tours: two Euler tours of a graph G are compatible if no pair of adjacent edges of G are consecutive in both tours.

Theorem (Jackson, 1991)

Let G be a 3-connected Eulerian graph. Then G has three pairwise compatible Euler tours.

Corollary

Let G be a 3-connected, 4-regular graph. Then $L(G)$ can be decomposed into three Hamiltonian cycles.

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If G is an Eulerian graph with $\delta(G) \geq 2k$, then G has a set of $2k-2$ pairwise compatible Euler tours.

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Let G be an Eulerian graph with $\delta(G) \geq 2k$. Then G has a set of $2k-1$ pairwise compatible Euler tours if and only if

$$(2k-1)(\omega(G^T)-1) \leq (2k-2)|T|$$

for all sets of disjoint transitions T in G.

CONJECTURE (KOTZIG, 1979)

K_{2k+1} *has* $(2k-1)$ *pairwise compatible Euler tours.*

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Theorem (Jackson, Wormald, 1990)

If G is an Eulerian graph with $\delta(G) \geq 2k$, then G has a set of k pairwise compatible Euler tours.

Theorem (Fleischner, Hilton, Jackson, 1990)

Let G be a connected Eulerian graph other than a cycle and such that the blocks of G are the cycles of G. If $\delta(G) \geq 2k$, then G has $2k-2$ pairwise compatible Euler tours.

Edge-disjoint Hamiltonian cycles in $L(G)$

Theorem (Bai, He, Li, Yang, 2015)

If a graph G is 4k-edge-connected, then there are k edge-disjoint Hamiltonian cycles in $L(G)$.

Theorem (Bai, He, Li, Yang, 2016)

If $L(G)$ is Hamiltonian, then there exist at least $max\{1, \lfloor \frac{1}{8}\delta(G) - \frac{3}{4} \rfloor\}$ edge-disjoint Hamiltonian cycles in $L(G)$.

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Robust Hamilton-connectedness of $L(G)$

Theorem (He, Yang, 2017)

Given a graph G, if $L(G)$ is Hamiltonian-connected, then every $SL(G) \in \mathcal{SL}(G)$ is also Hamiltonian-connected.

Corollary

If $L(G)$ is Hamiltonian-connected, then there exist at least max$\{1, \lfloor \frac{1}{8}\delta(G) \rfloor - 1\}$ edge-disjoint Hamiltonian paths between any two vertices in $L(G)$.

Further Researches

- More edges can be deleted to maintain the Hamiltonicity of line graphs?
- More edge-disjoint Hamiltonian cycles in a Hamiltonian line graph?
- For digraphs?

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CONJECTURE (FLEISCHNER, JACKSON, 1990)

If D is an Eulerian digraph with $\delta(D) \geq 2k$, then G has a set of $k-2$ pairwise compatible Euler tours.

THEOREM (FLEISCHNER, JACKSON, 1990)

If D is an Eulerian digraph with $\delta(D) \geq 2k$, then G has a set of $\lfloor \frac{k}{2} \rfloor$ pairwise compatible Euler tours.

- A Euler tour in a digraph $D \Leftrightarrow$ A Hamiltonian cycle in the line digraph $L(D)$.

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Thank you!