Extremal problems on digraphs

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Based on joint work with

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Background: Turán problems

- Let $\mathscr{F}$ be a family of graphs.

  $\mathscr{F}$-free graphs: graphs containing no subgraph from $\mathscr{F}$.

  $ex(n, \mathscr{F})$: the maximum number of edges in an $\mathscr{F}$-free graph with n vertices.

  $Ex(n, \mathscr{F})$: the $\mathscr{F}$-free graphs with n vertices that have $ex(n, \mathscr{F})$ edges.

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Turán problems

Given a family $\mathscr{F}$ of graphs, determine $ex(n, \mathscr{F})$ and $Ex(n, \mathscr{F})$.

Turán problems: complete graphs

Theorem (Mantel, 1907)

The number of edges in a triangle-free graph is at most $\lfloor n^2/4 \rfloor$*. Furthermore, the only triangle-free graph with* $\lfloor n^2/4 \rfloor$ *edges is the complete bipartite graph* $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

Theorem (Turán , 1941)

$ex(n, K_{t+1})$ *is attained at G if and only if G is a complete t-partite graph of order n with almost equal partite.*

The chromatic number $\mathcal{X}(G)$: the smallest natural number c such that the vertices of G can be coloured with c colours and no two vertices of the same colour are adjacent.

Theorem (Erdős-Stone-Simonovits)

$ex(n, H) = \frac{n^2}{2}\left[1 - \frac{1}{\mathcal{X}(H)-1} \pm o(1)\right].$

Turán problems: cycles

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Theorem (Füredi, 1996)

Let $q > 13$ be an integer. Then

$$ex(q^2 + q + 1, C_4) \leq \frac{1}{2}q(q+1)^2$$

with equality if q is a power of a prime number.

Theorem (Brown, Harary, 1969)

Let $\overrightarrow{K}_r$ be the complete digraph on r vertices and T_r be an orientation of K_r. Then

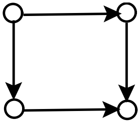
$$ex(n, \overrightarrow{K}_r) = \binom{n}{2} + ex(n, K_r)$$

and

$$ex(n, \overrightarrow{T}_r) = 2ex(n, K_r).$$

Turán problems on digraphs: orientation of C_4

- $P_{2,2}$:



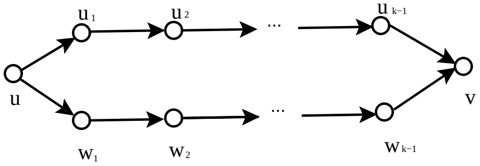
Theorem (H., Lyu, 2019+)

For strict digraphs, we have

$$ex(n, P_{2,2}) = \begin{cases} \frac{n^2+4n-1}{4}, & \text{if } n \text{ is odd;} \\ \frac{n^2+4n}{4}, & \text{if } \frac{n}{2} \text{ is even;} \\ \frac{n^2+4n-4}{4}, & \text{if } \frac{n}{2} \text{ is odd.} \end{cases}$$

Turán problems on digraphs

- Digraphs here allow loops but do not allow multiple arcs.
- Denote by $\mathscr{F}_k$ the family of digraphs consisting of two different walks of length k with the same initial vertex and the same terminal vertex, which has the form



Turán problems on digraphs

Problem 1 (Turán type version)

Let $n, k \geq 2$ be positive integers. Determine $ex(n, \mathscr{F}_k)$ and $Ex(n, \mathscr{F}_k)$.

Matrices and digraphs

- Digraph of a matrix $A \in M_n\{0,1\}$: $D(A) = (V, E)$ with vertex set $V = \{1, \ldots, n\}$ and arc set $E = \{(i,j) : a_{ij} = 1\}$.

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- Adjacency matrix of a digraph $D = (V, E)$ with $V = \{1, \ldots, n\}$: $A(D) = (a_{ij})_{n\times n} \in M_n\{0,1\}$ with $a_{ij} = 1$ if and only if $(i,j) \in E$.

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- $A^k \in M_n\{0,1\}$ if and only if there is at most one directed walk of length k from i to j in $D(A)$ for every pair of vertices i, j.

Matrix version of Problem 1

Problem 1 (Zhan, 2007)

Let $n, k \geq 2$ be positive integers. Determine the maximum number of 1's in an $n \times n$ 0-1 matrices A such that A^k is a 0-1 matrices. Characterize the matrices that attain the maximum number.

Solution to Problem 1: the case $k = 2$

Theorem 2 (Wu, 2010)

$$ex(n, \mathscr{F}_2) = \begin{cases} (n^2 + 4n - 1)/4, & \textit{if } n \textit{ is odd}, \\ (n^2 + 4n - 4)/4, & \textit{if } n \textit{ is even and } n \neq 4, \\ 8, & \textit{if } n = 4 \end{cases}$$

Solution to Problem 1: the case $n \leq k+1$

Theorem 3 (H., Zhan, 2011)

Let n, k be given integers with $n \geq 5$ and $n \leq k+1$. Then

$$ex(n, \mathscr{F}_k) = n(n-1)/2.$$

Moreover, $D \in Ex(n, \mathscr{F}_k)$ if and only if D is the transitive tournament of order n.

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- Let $n \geq 4$ and let D be a digraph of order n. If all D's induced subgraphs of order $n - 1$ are transitive tournaments, then D is a transitive tournament.

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- The result holds for $n = 5$.

Proof of Theorem 3

- Use induction on n. Suppose D has d loops.

  Counting the arcs of D in two ways:

$$\begin{aligned}(n-2)[f(D)-d]+(n-1)d &= \sum_{i=1}^{n} f(D-i) \\ &\leq n\frac{(n-1)(n-2)}{2}\end{aligned}$$

  which leads to

$$f(D) \leq n(n-1)/2 - d/(n-2) \leq n(n-1)/2.$$

Solution to Problem 1: the cases $n = k + 2$ and $n = k + 3$

Theorem 4 (H., Zhan, 2011)

If $n \geq 6$, *then*

$$ex(n, \mathscr{F}_{n-2}) = \frac{n(n-1)}{2} - 1.$$

If $n \geq 7$, *then*

$$ex(n, \mathscr{F}_{n-3}) = \frac{n(n-1)}{2} - 2.$$

Solution to Problem 1: Other cases

- Question: Let $k \geq 3$. For sufficiently large n,

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Theorem 5 (H., Lyu, 2019)

For $k \geq 4$ and $n = k + 4$, $ex(n, \mathscr{F}_k) = \frac{n(n-1)}{2} - 4$.

Solution to Problem 1: : the case $k \geq 4$

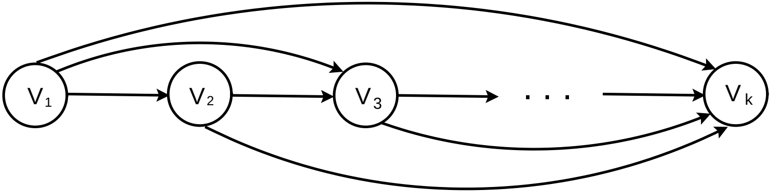
Theorem 6 (H., Lyu, Qiao, 2019)

Let $n = sk + t$ with s, k, t being nonnegative integers such that $t < k$. If $k \geq 4$ and $n \geq k + 4$, then

$$ex(n, \mathscr{F}_k) = \binom{n}{2} - \binom{s}{2}k - st.$$

Moreover, if $k \geq 5$ and $n \geq k + 5$, then a digraph D is in $Ex(n, \mathscr{F}_k)$ if and only if D is a balanced k-partite transitive tournament.

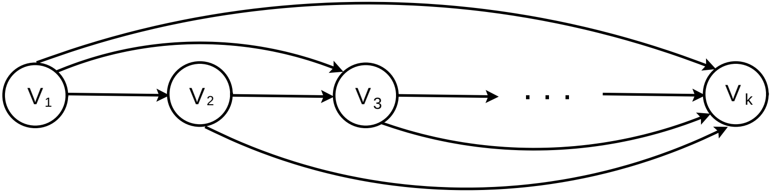
Balanced k-partite transitive tournament



- Let $\sum_{i=1}^{k} |V_i| = n = sk + t$ with $t < k$.

$$|V_i| = s \quad \text{or} \quad s+1 \quad \text{for} \quad i = 1, 2, \ldots, k.$$

Balanced k-partite transitive tournament



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$$|V_i| = s \quad \text{or} \quad s+1 \quad \text{for} \quad i = 1, 2, \ldots, k.$$

- For any $x \in V_i$ and $y \in V_j$, $x \to y$ if and only if $i < j$.

Sketch of proof for Theorem 6

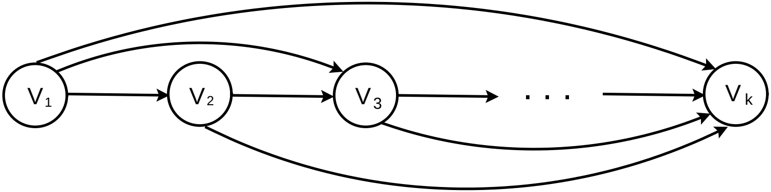

Figure: $Ex(k, k+2)$ for $k \geq 4$

- $|V_2| = \cdots = |V_{k-1}| = 1$,
- $(|V_1|, |V_k|) = (1, 2)$ or $(2, 1)$.

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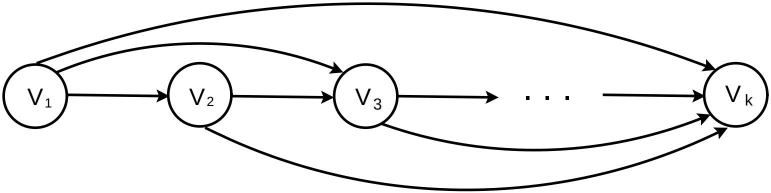

Figure: $Ex(k, k+3)$ for $k \geq 4$

- $|V_2| = \cdots = |V_{k-1}| = 1$, $|V_1| = |V_k| = 2$.

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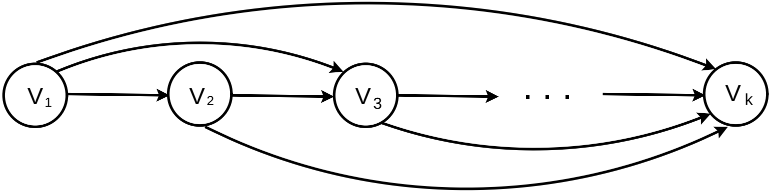

Figure: $Ex(k, k+4)$ for $k \geq 5$

- $|V_2| = \cdots = |V_{k-1}| = 1$,
- $(|V_1|, |V_k|) = (2, 3)$ or $(3, 2)$.

Sketch of proof for Theorem 6

- For $k \geq 5$ and $n = k + 5$, $ex(n, \mathscr{F}_k) = \binom{n}{2} - 5$. Moreover, a digraph D is in $Ex(n, \mathscr{F}_k)$ if and only if D is a balanced k-partite blow-up transitive tournament.

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- Use induction on n

Unsolved cases of Problem 1

- $ex(n, \mathscr{F}_3)$
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- $Ex(n, \mathscr{F}_4)$

Thank you!