The Degree Sequence Related Problems

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- Forcible Sequences
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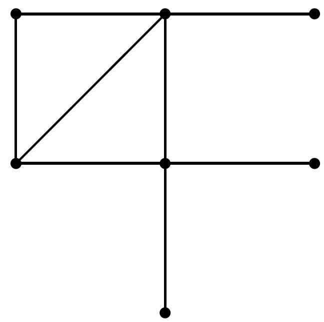
Degree Sequences Realization

- Example



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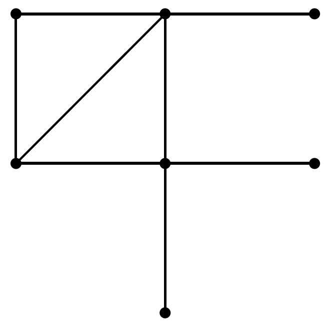

- $d = (4, 4, 3, 2, 1, 1, 1)$

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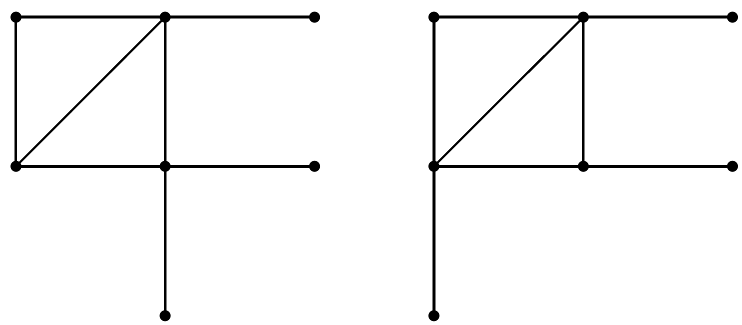

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- Theorem (Bollobas, DM 1979) If for any $1 \leq i \leq \min\{n/2 - 1, d_n\}$, $\sum_{j=1}^{i}(d_i + d_{n-i}) \geq in - 1$, then $\kappa'(G) = \delta(G) = d_n$.

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- Let D be a (strict) digraph, $d^+ = (d_1^+, d_2^+, ..., d_n^+)$ and $d^- = (d_1^-, d_2^-, ..., d_n^-)$ be the out- and in-degree sequences of D with $d_1^+ \leq d_2^+ \leq ... \leq d_n^+$ and $d_1^- \leq d_2^- \leq ... \leq d_n^-$. (They are called a pair of digraphic sequences.)

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- Open Problem. Characterize forcible sequences for k-connected (k-edge-connected) graphs.

Degree Sequences Realization Problem

- **Problem** An n processor network has n processors $v_1, v_2, ..., v_n$ such that each v_i has a given number d_i of connections. It is expected to design such a network so that it has a certain level of strength or reliability.

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- A potential degree sequence for a graphical property P (or a potential P-sequence) is a degree sequence d such that there exists one $G \in (d)$ that has property P.

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- Major Tool: Given $G \in (d)$, do
  (i) Find a matching $e_1, e_2 \in E(G)$ and a matching f_1, f_2 in G^c such that e_1, e_2, f_1, f_2 form a 4-cycle in K_n.
  (ii) Switching: $G' := G - \{e_1, e_2\} + \{f_1, f_2\} \in (d)$.
  (iii) Compare to see if G' is more favorable than G.

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- Problem. For a given integer $k > 0$, can we characterize graphic sequences $d = (d_1, d_2, \cdots, d_n)$ which has a realization with k-edge-disjoint spanning trees?

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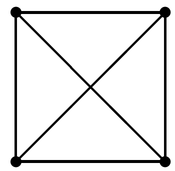
- Example



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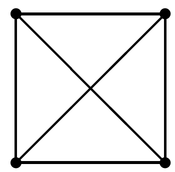
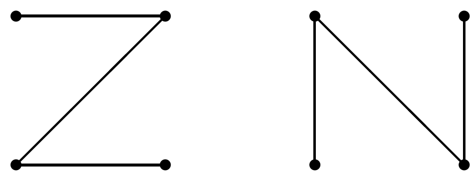


- Two edge-disjoint spanning trees



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- Theorem (Nash-Williams, Tutte [J. London Math. Soc. (1961)]) For a connected graph G, $\tau(G) \geq k$ if and only if $\forall X \subseteq E(G)$, $|X| \geq k(\omega(G - X) - 1)$.

Spanning Trees Packing and Covering

- Theorem (Liang, Li, HJL DAM 2010) Let $d = (d_1, \cdots, d_n)$ be a nonincreasing graphic sequence. Then d has a realization G with $\tau(G) \geq k$ if and only if either $n = 1$ and $d_1 = 0$ or $n > 1$ and each of the following statements holds
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- Necessity: Let $T_1, T_2, ..., T_k$ be disjoint spanning trees of G. Then

$$\begin{aligned}
\sum_{i=1}^{n} d_i &= 2|E(G)| \\
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- Proving the sufficiency is more involved.

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- Theorem (Nash-Williams [J. London Math. Soc. (1964)]) For a connected graph G, $a_1(G) \leq k$ if and only if $\forall X \subseteq E(G)$, $|X| \leq k(|V(G[X])| - \omega(G[X]))$.

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- Theorem (Liu, Zhang, Zhang and HJL, DAM 2015) Let $k_2 \ge k_1 \ge 0$ and $n > 1$ be integers. Let $d = (d_1, d_2, \cdots, d_n)$ with $d_1 \ge d_2 \ge \cdots \ge d_n$ be a graphic sequence and let $I = \{i : d_i < k_2\}$. Then there exists a graph $G \in (d)$ such that $k_2 \ge a(G) \ge \tau(G) \ge k_1$ if and only if each of the following holds.
  (i) $d_n \ge k_1$.
  (ii) $2k_2(n - |I| - 1) + 2\sum_{i \in I} d_i \ge \sum_{i=1}^n d_i \ge 2k_1(n-1)$.

Spanning Trees Packing and Covering

- Corollary (Liu, Zhang, Zhang and HJL, DAM 2015) Let $n \geq 2$ and $k > 0$ be integers. For a graphic sequence $d = (d_1, d_2, \cdots, d_n)$ with $d_1 \geq d_2 \geq \cdots \geq d_n$, the following are equivalent.
  (i) There exists a d-realization G such that $a(G) \leq k$.
  (ii) $\sum_{i=1}^{n} d_i \leq 2k(n - |I| - 1) + 2\sum_{i \in I} d_i$, where $I = \{i : \ d_i < k\}$.

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- Open Problem: What are the forcible degree sequence condition for Spanning Trees Packing and Covering?

Supereulerian and line-hamiltonian

Realizations

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- G is eulerian if G is connected with $O(G) = \emptyset$.
- A graph G is supereulerian if G contains a spanning eulerian subgraphs.

Supereulerian and line-hamiltonian Realizations

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- Theorem (Y. Tian, J, Meng, Z. Zhang and HJL, DM 2014) Let $d = (d_1, d_2, \cdots, d_n)$ with $d_1 \geq d_2 \geq \cdots \geq d_n = 1$ be a graphic sequence. Then there exists a graph $G \in (d)$ such that G is super edge-connected if and only if one of the following holds.
  (i) $d_1 = n - 1$ and $\sum_{i=1}^{n} d_i = 2(n-1)$, or
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- If $t < n$, then $\{v_{n-t+1}, ..., v_n\}$ is an independent set and so $\displaystyle\sum_{i=1}^{n-t} d_i \geq \sum_{i=n-t+1}^{n} d_i = 2t$. As $2|E(G - \{v_{n-t+1}, ..., v_n\})| = \displaystyle\sum_{i=1}^{n-t} d_i - 2t$, we have $d_1 \leq t + \frac{1}{2}\left(\sum_{i=1}^{n-t} d_i - 2t\right)$.

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- Theorem (Y. Tian, J, Meng, Z. Zhang and HJL, DM 2014) Every graphic sequence $d = (d_1, d_2, \cdots, d_n)$ with $d_1 \geq d_2 \geq \cdots \geq d_n \geq 3$ has a super edge-connected realization.

Nowhere-zero Flows and Modulo Orientation Realizations

- Definition Given an integer $k > 0$ and a graph G with a fixed orientation D. A function $f : E(G) \to Z_k - \{0\}$ is a nowhere-zero Z_k-flow if at every vertex v, the flow-in amount amount at v equals to the flow-out amount amount at v under f.

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- Theorem (Fulkerson (1960)-Ryser (1963)) Let $d = \{(d_1^+, d_1^-), \ldots, (d_n^+, d_n^-)\}$ be a sequence of non-negative integer pairs with $d_1^+ \geq \ldots \geq d_n^+$. Then d is strict digraphic if and only if each of the following holds:
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- Open Problem. For $k > 1$, characterize forcible digraphic sequences for strong k-connectedness (arc k-connectedness).

Strong digraph realizations

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- Theorem (Y. Hong, Q. Liu, HJL JGT 2017) $\mathbf{d}$ has a strong strict $\mathbf{d}$-realization if and only if both of the following hold.
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- Related Problem. For any given integer $k > 0$, is there an integer $f_1(k)$ such that every graph G with $\tau(G) \geq f_1(k)$ has k edge-disjoint spanning trees $T_1, T_2, \cdots, T_k$ such that $\Delta(T_i) \leq \lceil \Delta(G)/k \rceil$?

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Thank you!