Multicoloured Ramsey number of the path of length four

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Joint work in progress with Bojan Mohar[2] and Yongtang Shi[3]

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Introduction

For graphs $H_1, \ldots, H_m$, let $R(H_1, \ldots, H_m)$ denote the m-coloured Ramsey number of $H_1, \ldots, H_m$. That is, $R(H_1, \ldots, H_m)$ is the minimum integer n s.t. every m-edge-colouring of K_n has a copy of H_i in colour i, for some i.

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- Dynamic survey by Radziszowski in Electron. J. Combin., DS1. Most recent version: March 2017.

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From Radziszowski's EJC survey:

5.1. Paths

$R(P_m, P_n) = n + \lfloor m/2 \rfloor - 1$ for all $n \geq m \geq 2$ [GeGy]

Classification of $R(P_m, P_n)$-critical graphs [Hook]

Stripes mP_2 [CocL1, CocL2, Lor]

Disjoint unions of paths (also called linear forests) [BuRo2, FS2]

Monotone paths [CaYZ], ordered path powers [Mub2]

6.4.1. Three-color path and path-cycle cases

(a) $R(P_m, P_n, P_k) = m + \lfloor n/2 \rfloor + \lfloor k/2 \rfloor - 2$ for $m \geq 6(n+k)^2$ [FS2],
the equality holds asymptotically for $m \geq n \geq k$ with an extra term $o(m)$ [FiŁu1],
extensions of the range of m, n, k for which (a) holds were obtained in [Biel3].

(b) $R(P_3, P_m, P_n) = m + \lfloor n/2 \rfloor - 1$ for $m \geq n$ and $(m, n) \neq (3,3), (4,3)$ [MaORS2].

(c) $R_3(P_3) = 5$ [Ea1], $R_3(P_4) = 6$ [Ir],
$R(P_m, P_n, P_k) = 5$ for other $m{-}n{-}k$ combinations with $3 \leq m, n, k \leq 4$ [ArKM],
$R_3(P_5) = 9$ [YR1], $R_3(P_6) = 10$ [YR1], and $R_3(P_7) = 13$ [YY],
$R_3(P_8) = 14$, $R_3(P_9) = 17$ [DyDR].

(d) $R(P_4, P_4, P_{2n}) = 2n + 2$ for $n \geq 2$,
$R(P_5, P_5, P_5) = R(P_5, P_5, P_6) = 9$,
$R(P_5, P_5, P_n) = n + 2$ for $n \geq 7$,
$R(P_5, P_6, P_n) = R(P_4, P_6, P_n) = n + 3$ for $n \geq 6$,
$R(P_6, P_6, P_{2n}) = R(P_4, P_8, P_{2n}) = 2n + 4$ for $n \geq 14$ [OmRa1].

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$R_m(P_2) = m + 1 + (m \bmod 2)$ *for all* $m \geq 1$.

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Theorem 2 (Various authors)

For $m \geq 1$,

$$R_m(P_3) = \begin{cases} 2m+1 & \text{if } m \equiv 0,2 \pmod 3,\ m \neq 3, \\ 2m+2 & \text{if } m \equiv 1 \pmod 3, \\ 6 & \text{if } m = 3. \end{cases}$$

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6.4.2. More colors

(a) $R_k(P_3) = k+1+(k \bmod 2)$, $R_k(2P_2) = k+3$ for all $k \geq 1$ [Ir].

(b) $R_k(P_4) = 2k + c_k$ for all k and some $0 \leq c_k \leq 2$. If k is not divisible by 3 then $c_k = 3 - k \bmod 3$ [Ir]. Wallis [Wall] showed $R_6(P_4) = 13$, which already implied $R_{3t}(P_4) = 6t+1$, for all $t \geq 2$. Independently, the case $R_k(P_4)$ for $k \neq 3^m$ was completed by Lindström in [Lind], and later Bierbrauer proved $R_{3^m}(P_4) = 2(3^m) + 1$ for all

$m > 1$. $R_3(P_4) = 6$ [Ir].

(c) $R_k(P_n) \leq (k - c_k)n$ for some $c_k > 0$, for all fixed $k \geq 2$ and large n [Sár2, DavJR].

(d) Formula for $R(P_{n_1}, \ldots, P_{n_k})$ for large n_1 [FS2], and some extensions [Biel3]. Conjectures about $R(P_{n_1}, \ldots, P_{n_k})$ when all or all but one of n_i's are even [OmRa1].

Ramsey number of P_4

Theorem 3 (L., Mohar, Shi, 2019+)

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$$R_m(P_4) = \begin{cases} 3m+1 & \text{if } m \equiv 0 \pmod 4,\ m \neq 4, \\ 3m+2 & \text{if } m \equiv 1 \pmod 4, \\ 3m & \text{if } m \equiv 2,3 \pmod 4, \\ 11 & \text{if } m = 4. \end{cases}$$

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Definition

The Turán function of H is

$$\text{ex}(n, H) = \max\{e(G) : |V(G)| = n \text{ and } G \not\supset H\}.$$

Theorem 4 (Faudree and Schelp, 1975)

Let $n = 4a + b$, where $a \geq 0$ and $0 \leq b \leq 3$. Then $\mathrm{ex}(n, P_4) = 6a + \binom{b}{2}$. The unique extremal graph is $aK_4 \,\dot{\cup}\, K_b$.

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Take any m-edge-colouring of K_n. Thm. 4 $\Rightarrow$ If $m \equiv 0, 1, 2 \pmod 4$, and $n = 3m+1, 3m+2, 3m$, then $\lceil \binom{n}{2}/m \rceil > \mathrm{ex}(n, P_4)$.

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Clearly, resolvable $\Rightarrow v \equiv 0 \pmod{k}$.

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Theorem 5 (Lamken et al. (1998); Abel et al. (2007))

For $v \equiv 0 \pmod 4$ with $v \neq 12$, there exists an optimal resolvable $(2,4,v)$-covering design, except possibly for $v \in \{108, 116, 132, 156, 204, 212\}$. Moreover, the number of parallel classes in a resolution is

$$\begin{cases} \dfrac{v}{3} & \text{if } v \equiv 0 \pmod{12}, \\ \dfrac{v-1}{3} & \text{if } v \equiv 4 \pmod{12}, \\ \dfrac{v+1}{3} & \text{if } v \equiv 8 \pmod{12}. \end{cases}$$

Theorem 6 (Brouwer (1979); Ge et al. (2005))

For $v \equiv 0 \pmod 4$ *with* $v \neq 8, 12, 20$, *there exists an optimal resolvable* $(2, 4, v)$*-packing design, except possibly for 18 values of* v, *none of which is* $108, 116, 132, 156, 204, 212$. *Moreover, the number of parallel classes in a resolution is*

$$\begin{cases} \dfrac{v-3}{3} & \text{if } v \equiv 0 \pmod{12}, \text{ and } R = \frac{v}{3}K_3, \\ \dfrac{v-1}{3} & \text{if } v \equiv 4 \pmod{12}, \text{ and } R = E_v, \\ \dfrac{v-2}{3} & \text{if } v \equiv 8 \pmod{12}, \text{ and } R = \frac{v}{2}K_2. \end{cases}$$

Now we prove for $m \neq 4$:

$$R_m(P_4) \geq \begin{cases} 3m+1 & \text{if } m \equiv 0 \pmod{4}, \\ 3m+2 & \text{if } m \equiv 1 \pmod{4}, \\ 3m & \text{if } m \equiv 2,3 \pmod{4}. \end{cases}$$

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Case $m \equiv 0 \pmod 4$: Same argument $\Rightarrow R_m(P_4) \geq 3m + 1$, except for $m \in \{36, 44, 52, 68\}$.

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Case $m \in \{36, 44, 52, 68\}$: Same argument $\Rightarrow \exists$ $(m - 1)$-edge-colouring of $K_{3m} - E(mK_3)$ with no monochromatic $P_4 \Rightarrow R_m(P_4) \geq 3m + 1$.

Finally, $R_4(P_4) \geq 11$:

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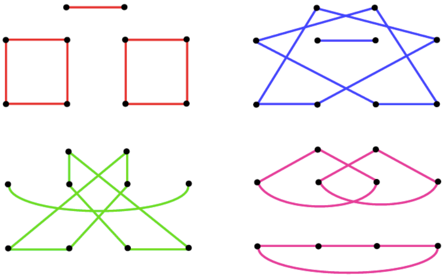

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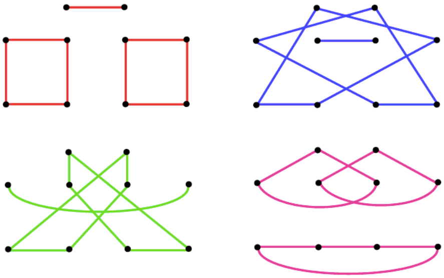

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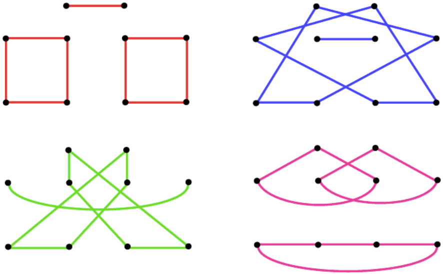

$R_4(P_4) \leq 11$: Assume $\exists$ 4-edge-colouring of K_{11} with no monochromatic P_4. Let $G_1, \ldots, G_4$ be the four coloured graphs on $V(K_{11})$, $e(G_1) \geq \cdots \geq e(G_4)$. Thm. 4 $\Rightarrow e(G_1) \leq 15$.

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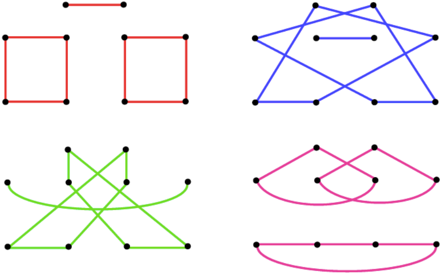

$R_4(P_4) \leq 11$: Assume $\exists$ 4-edge-colouring of K_{11} with no monochromatic P_4. Let $G_1, \ldots, G_4$ be the four coloured graphs on $V(K_{11})$, $e(G_1) \geq \cdots \geq e(G_4)$. Thm. 4 $\Rightarrow e(G_1) \leq 15$.

- $e(G_1) = 15 \Rightarrow G_1 = K_4 \,\dot\cup\, K_4 \,\dot\cup\, K_3$, and $e(G_2) \geq 14 \Rightarrow K_4 \subset G_2$. Impossible.

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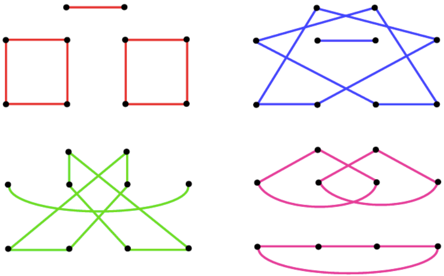

$R_4(P_4) \leq 11$: Assume $\exists$ 4-edge-colouring of K_{11} with no monochromatic P_4. Let $G_1, \ldots, G_4$ be the four coloured graphs on $V(K_{11})$, $e(G_1) \geq \cdots \geq e(G_4)$. Thm. 4 $\Rightarrow e(G_1) \leq 15$.

- $e(G_1) = 15 \Rightarrow G_1 = K_4 \dot\cup K_4 \dot\cup K_3$, and $e(G_2) \geq 14 \Rightarrow K_4 \subset G_2$. Impossible.
- $e(G_1) = 14 \Rightarrow e(G_2) = e(G_3) = 14$. Then $G_1 = K_4 \dot\cup K_4^- \dot\cup K_3$ or $K_4 \dot\cup K_4 \dot\cup P_2$, and $K_4 \subset G_2, G_3$. Impossible.

Thank you!