A Note on Systems of Equations of Sum of Equal Powers*

Rui Gao[1] Jun Liao[1] Heguo Liu[1] Huan Xiong[2] Xingzhong Xu[1]

[1] School of Mathematics and Statistics, Hubei University, Wuhan, P.R. China
ruigao131@hotmail.com;{jliao,ghliu}@hubu.edu.cn;xuxingzhong407@126.com

[2] I.R.M.A., UMR 7501, Université de Strasbourg et CNRS, 67084 Strasbourg, France
xiong@math.unistra.fr

Abstract

In this paper, we study the solutions of the following system of equations of sum of equal powers:

$$\begin{cases} x_1^{k_1} + x_2^{k_1} + \cdots + x_n^{k_1} = a \\ x_1^{k_2} + x_2^{k_2} + \cdots + x_n^{k_2} = a \\ \qquad\qquad \vdots \\ x_1^{k_n} + x_2^{k_n} + \cdots + x_n^{k_n} = a, \end{cases}$$

where $0 < k_1 < k_2 < \cdots < k_n$, $k_1, k_2, \cdots, k_n$ are natural numbers, and $a \in \{0, 1, n\}$. Our main theorem generalizes several known results.

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1 Introduction and Main Results

Symmetric function theory is an important research field in algebraic combinatorics. It has wide applications in many areas such as representation theory of finite group, algebraic geometry, Lie algebra, polynomial equation theory, special functions and theoretical physics, see [1, 2, 3, 4]. For the indeterminates $x_1, x_2, \cdots, x_n$, we have

$$s_k - \sigma_1 s_{k-1} + \sigma_2 s_{k-2} - \cdots + (-1)^{k-1}\sigma_{k-1}s_1 + (-1)^k k\sigma_k = 0 \quad (\text{for} \quad 1 \leqslant k \leqslant n); \qquad (1)$$

$$s_k - \sigma_1 s_{k-1} + \sigma_2 s_{k-2} - \cdots + (-1)^n \sigma_n s_{k-n} = 0 \quad (\text{for} \quad k > n), \qquad (2)$$

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where $\sigma_i = \sum\limits_{1\leqslant l_1<l_2<\cdots<l_i\leqslant n} x_{l_1}x_{l_2}\cdots x_{l_i}, i = 1,2,\cdots,n$ and $s_j = \sum\limits_{l=1}^{n} x_l^j, j \in \mathbb{Z}^+$. The equations above are known as Newton's identities. There are many ways to prove Newton's identities, see for example [5, 6, 7, 8].

It is well known that the following system of equations of sum of equal powers

$$\begin{cases} x_1 + x_2 + \cdots + x_n = p_1 \\ x_1^2 + x_2^2 + \cdots + x_n^2 = p_2 \\ \qquad\qquad \vdots \\ x_1^n + x_2^n + \cdots + x_n^n = p_n \end{cases} \tag{3}$$

can be solved by reducing it to a single-variable n-th polynomial equation using Newton's formulas, where $p_i \in \mathbb{C}, i = 1,2,\cdots,n$, see [3, 9]. For some special arrays $\{p_1,p_2,\cdots,p_n\}$, the system of equations (3) can be solved explicitly. In particular,

Case 1 $p_1 = p_2 = \cdots = p_n = 0$.

We have $\sigma_i = 0, 1 \leqslant i \leqslant n$. Thus $x_1,x_2,\cdots,x_n$ are the roots of $x^n = 0$. So (3) has a unique solution $(0,0,\cdots,0)$.

Case 2 $p_1 = a, p_2 = a^2, \cdots, p_n = a^n$.

By Case 1, we may assume that $a \neq 0$. Let $y_1 = \frac{x_1}{a}, y_2 = \frac{x_2}{a}, \cdots, y_n = \frac{x_n}{a}$. Then (3) reduces to the case when $a = 1$. Hence we have $\sigma_1 = 1, \sigma_i = 0, 2 \leqslant i \leqslant n$. Thus $y_1,y_2,\cdots,y_n$ are the roots of $x^n - x^{n-1} = x^{n-1}(x-1) = 0$. So (3) has a unique solution (ignoring order) $(a,0,\cdots,0)$.

Case 3 $p_1 = p_2 = \cdots = p_n = n$.

We have $\sigma_k = C_n^k, k = 1,2,\cdots,n$. Thus $x_1,x_2,\cdots,x_n$ are the roots of $x^n - C_n^1x^{n-1} + C_n^2x^{n-2} - \cdots + (-1)^nC_n^n = (x-1)^n = 0$. So (3) has a unique solution $(1,1,\cdots,1)$.

Throughout this note, we denote ω the n-th primitive root of 1 and $\mathbb{N}^+$ or $\mathbb{Z}^+$ the set of positive integers. We mainly consider the following system of equations of sum of equal powers:

$$\begin{cases} x_1^{k_1} + x_2^{k_1} + \cdots + x_n^{k_1} = a \\ x_1^{k_2} + x_2^{k_2} + \cdots + x_n^{k_2} = a \\ \qquad\qquad \vdots \\ x_1^{k_n} + x_2^{k_n} + \cdots + x_n^{k_n} = a \end{cases} \tag{4}$$

where $0 < k_1 < k_2 < \cdots < k_n$, $k_1,k_2,\cdots,k_n \in \mathbb{N}$, $a \in \{0,1,n\}$. We derive the following results:

Theorem 1. *Let $m,n \in \mathbb{N}$, $m \geqslant n \geqslant 2$. Then the system of equations of sum of equal powers*

$$\begin{cases} x_1 + x_2 + \cdots + x_n = n \\ x_1^2 + x_2^2 + \cdots + x_n^2 = n \\ \qquad\qquad \vdots \\ x_1^{n-1} + x_2^{n-1} + \cdots + x_n^{n-1} = n \\ x_1^m + x_2^m + \cdots + x_n^m = n \end{cases} \tag{5}$$

(i) has a unique solution $(1, 1, \cdots, 1)$ *if and only if* $n \leqslant m < 2n$.

(ii) if $kn \leqslant m < (k+1)n, k = 2, 3, \cdots$, *the solutions are* $(1 + t\omega, 1 + t\omega^2, \cdots, 1 + t\omega^n)$, *where* t *are the roots of* $f(x) = x^n \sum_{i=0}^{k-1} C_m^{(i+1)n} (x^n)^i$.

Theorem 2. *Let* $m, n \in \mathbb{N}$, $m \geqslant n \geqslant 2$. *Then the system of equations of sum of equal powers*

$$\begin{cases} x_1 + x_2 + \cdots + x_n = 1 \\ x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \\ \qquad\qquad \vdots \\ x_1^{n-1} + x_2^{n-1} + \cdots + x_n^{n-1} = 1 \\ x_1^m + x_2^m + \cdots + x_n^m = 1 \end{cases}$$

has a unique solution (ignoring order)$(1, 0, \cdots, 0)$ *if and only if* $n \leqslant m < 2n$.

Theorem 3. *Let* $m, n \in \mathbb{N}$, $m \geqslant n \geqslant 2$. *Then the system of equations of sum of equal powers*

$$\begin{cases} x_1 + x_2 + \cdots + x_n = 0 \\ x_1^2 + x_2^2 + \cdots + x_n^2 = 0 \\ \qquad\qquad \vdots \\ x_1^{n-1} + x_2^{n-1} + \cdots + x_n^{n-1} = 0 \\ x_1^m + x_2^m + \cdots + x_n^m = 0 \end{cases}$$

has a unique solution $(0, 0, \cdots, 0)$ *if and only if* $m = kn$. *When* $kn < m < (k+1)n$, *the solutions are* $(a\omega, a\omega^2, \cdots, a\omega^n)$, *where* $a \in \mathbb{C}$ *can take any complex number.*

In addition, we study several other cases of the array $\{k_1, k_2, \cdots, k_n\}$ with unique solutions to (4) and derive the following results, which shows the complexity of solving (4).

Theorem 4. *Let* $m \in \mathbb{N}^+$. *Then the system of equations of sum of equal powers*

$$\begin{cases} x_1^m + x_2^m + \cdots + x_n^m = 0 \\ x_1^{m+1} + x_2^{m+1} + \cdots + x_n^{m+1} = 0 \\ \vdots \\ x_1^{m+n-2} + x_2^{m+n-2} + \cdots + x_n^{m+n-2} = 0 \\ x_1^{m+n-1} + x_2^{m+n-1} + \cdots + x_n^{m+n-1} = 0 \end{cases}$$

has a unique solution $(0, 0, \cdots, 0)$.

Theorem 5. *Let $m, n, k \in \mathbb{N}^+$, $m > n$. Let the system of equations of sum of equal powers be*

$$\begin{cases} x_1 + x_2 + \cdots + x_n = 0 \\ \vdots \\ x_1^k + x_2^k + \cdots + x_n^k = 0 \\ x_1^{k+2} + x_2^{k+2} + \cdots + x_n^{k+2} = 0 \\ \vdots \\ x_1^n + x_2^n + \cdots + x_n^n = 0 \\ x_1^m + x_2^m + \cdots + x_n^m = 0 \end{cases}$$

where $n = p(k+1) + r_1$, $1 \leqslant k \leqslant n-2$, $0 \leqslant r_1 \leqslant k$, and $p \in \mathbb{N}^+$. Let $m = q(k+1) + r_2$ where $0 \leqslant r_2 \leqslant k$, $q \geqslant p$, and $q \in \mathbb{N}^+$. Then,

(i) when $r_2 = 0$, that is $(k+1)|m$, the system of equations has a unique solution;

(ii) when $r_2 \neq 0$, the solutions are the roots of $x^{n-p(k+1)}(x^{p(k+1)} + cx^{(p-1)(k+1)} + \frac{c^2}{2!}x^{(p-2)(k+1)} + \cdots + \frac{c^{(p-1)}}{(p-1)!}x^{k+1} + \frac{c^p}{p!}) = 0$, where $c \in \mathbb{C}$ can take any complex number.

2 Proof of Theorem 1

To prove Theorem 1, we need the following lemma.

Lemma 6. *Let $kn \leqslant m < (k+1)n$, $k = 1, 2, \cdots$. Then,*

(i) $C_m^{kn} - C_{m-1}^{kn}C_n^1 + C_{m-2}^{kn}C_n^2 - \cdots + (-1)^{m-kn}C_{kn}^{kn}C_n^{m-kn} = C_{m-n}^{(k-1)n}$,

(ii) $C_m^{sn} - C_{m-1}^{sn}C_n^1 + C_{m-2}^{sn}C_n^2 - \cdots + (-1)^{n-1}C_{m-n+1}^{sn}C_n^{n-1} + (-1)^n C_{m-n}^{sn}C_n^n = C_{m-n}^{(s-1)n}$, where $s = 1, 2, \cdots, k-1$.

Proof. (i) Recall that $(1+x)^n = 1 + C_n^1 x + C_n^2 x^2 + \cdots + C_n^n x^n$. We consider the formal power series

$$\frac{1}{(1+x)^{kn+1}} = 1 - C_{kn+1}^{kn}x + C_{kn+2}^{kn}x^2 - C_{kn+3}^{kn}x^3 + \cdots + (-1)^t C_{kn+t}^{kn}x^t + \cdots,$$

$$\frac{1}{(1+x)^{(k-1)n+1}} = 1 - C_{(k-1)n+1}^{(k-1)n}x + C_{(k-1)n+2}^{(k-1)n}x^2 - \cdots + (-1)^s C_{(k-1)n+t}^{(k-1)n}x^t + \cdots.$$

Note that

$$\frac{1}{(1+x)^{(k-1)n+1}} = (1+x)^n \frac{1}{(1+x)^{kn+1}}.$$

Since $m - kn < n$, compare the coefficient of x^{m-kn}, we have

$$\begin{aligned}(-1)^{m-kn}C_{(k-1)n+m-kn}^{(k-1)n} =&(-1)^{m-kn}C_{kn+m-kn}^{kn}C_n^0 + (-1)^{m-kn-1}C_{kn+m-kn-1}^{kn}C_n^1 + \cdots \\ &- C_{kn+1}^{kn}C_n^{m-kn-1} + C_{kn}^{kn}C_n^{m-kn}.\end{aligned}$$

Multiply both sides by $(-1)^{m-kn}$, we get

$$C_m^{kn} - C_{m-1}^{kn}C_n^1 + C_{m-2}^{kn}C_n^2 - \cdots + (-1)^{m-kn}C_{kn}^{kn}C_n^{m-kn} = C_{m-n}^{(k-1)n}.$$

(ii) For $s = 1, 2, \cdots, k-1$, consider the formal power series

$$\frac{1}{(1+x)^{sn+1}} = 1 - C_{sn+1}^{sn}x + C_{sn+2}^{sn}x^2 - C_{sn+3}^{sn}x^3 + \cdots + (-1)^t C_{sn+t}^{sn}x^t + \cdots,$$

$$\frac{1}{(1+x)^{(s-1)n+1}} = 1 - C_{(s-1)n+1}^{(s-1)n}x + C_{(s-1)n+2}^{(s-1)n}x^2 - \cdots + (-1)^s C_{(s-1)n+t}^{(s-1)n}x^t + \cdots.$$

Note that

$$\frac{1}{(1+x)^{(s-1)n+1}} = (1+x)^n \frac{1}{(1+x)^{sn+1}}.$$

Since $m - sn \geqslant n$, compare the coefficient of x^{m-sn}, we have

$$\begin{aligned}(-1)^{m-sn}C_{(s-1)n+m-sn}^{(s-1)n} =& (-1)^{m-sn}C_{sn+m-sn}^{sn}C_n^0 + (-1)^{m-sn-1}C_{sn+m-sn-1}^{sn}C_n^1 + \cdots + \\ & (-1)^{m-sn-(n-1)}C_{sn+m-sn-(n-1)}^{sn}C_n^{n-1} + (-1)^{m-sn-n}C_{sn+m-sn-n}^{sn}.\end{aligned}$$

Multiply both sides by $(-1)^{m-sn}$, we have

$$C_{m-n}^{(s-1)n} = C_m^{sn}C_n^0 - C_{m-1}^{sn}C_n^1 + \cdots + (-1)^{n-1}C_{m-(n-1)}^{sn}C_n^{n-1} + (-1)^n C_{m-n}^{sn}C_n^n.$$

This completes the proof of the lemma. $\square$

Next, we give the proof of Theorem 1.

Proof of Theorem 1. (i) By (1) and induction on i, we obviously have $\sigma_i = C_n^i$, where $1 \leqslant i \leqslant n-1$. Furthermore, by (1) and (2), we have

$$s_k - s_{k-1}\sigma_1 + s_{k-2}\sigma_2 - \cdots + (-1)^{n-1}s_{k-n+1}\sigma_{n-1} + (-1)^n s_{k-n}\sigma_n = 0 \quad (\text{for } \ k \geqslant n), \quad (6)$$

where $s_0 = n$. So $s_n - n = (-1)^{n+1}n\sigma_n - n(-1)^{n-1}$. Let $y = (-1)^n(1-\sigma_n)$. Then $s_n - n = ny$. For $n \leqslant m < 2n$, we have $0 \leqslant m-n < n$ and therefore $s_{m-n} = n$. Next, we prove

$$s_m - n = C_m^n \cdot ny \quad (7)$$

by induction on m. The result is obviously true when $m = n$. Suppose that $s_m - n = C_m^n \cdot ny$ holds for $m \leqslant k$, where $n \leqslant k \leqslant 2n-2$. Assume that $m = k+1$. By (6), we have

$$s_{k+1} - n = (s_k - n)\sigma_1 - (s_{k-1} - n)\sigma_2 + \cdots + (-1)^{k+2-n}(s_n - n)\sigma_{k+1-n} + (s_n - n).$$

Since $n \leqslant k \leqslant 2n-2$, it follows that $k+1-n < n$. Then by induction,

$$\begin{aligned}s_{k+1} - n =& C_k^n C_n^1 \cdot ny - C_{k-1}^n C_n^2 \cdot ny + \cdots + (-1)^{k+2-n}C_n^{k+1-n} \cdot ny + ny \\ =& ny(C_k^n C_n^1 - C_{k-1}^n C_n^2 + \cdots + (-1)^{k+2-n}C_n^{k+1-n} + 1).\end{aligned}$$

Thus by Lemma 6, $C_k^nC_n^1 - C_{k-1}^nC_n^2 + \cdots + (-1)^{k+2-n}C_n^{k+1-n} + 1{=}C_{k+1}^n$. Then $s_{k+1} - n = C_{k+1}^n \cdot ny$ and thus (7) is true.

However, by the last equality of (5), we have $s_m = n$. Therefore, when $n \leqslant m < 2n$, by (7) we obtain $y = 0$ and thus $\sigma_n = 1$. In this case, $x_1, x_2, \cdots, x_n$ are the roots of $x^n - C_n^1x^{n-1} + C_n^2x^{n-2} - \cdots + (-1)^{n-1}C_n^1x + (-1)^n = (x-1)^n = 0$. Therefore, the system of equations has a unique solution $(1, 1, \cdots, 1)$. In fact, we have $0 = s_m - n = C_m^n \cdot ny = C_m^n(s_n - n)$. For $n \leqslant m < 2n$, the system of equations is equivalent to

$$\begin{cases} x_1 + x_2 + \cdots + x_n = n \\ x_1^2 + x_2^2 + \cdots + x_n^2 = n \\ \qquad\qquad \vdots \\ x_1^n + x_2^n + \cdots + x_n^n = n. \end{cases}$$

For $m \geqslant 2n$, assume that $kn \leqslant m < (k+1)n, k = 2, 3, \cdots$. We prove

$$s_m - n = ny\sum_{i=0}^{k-1} C_m^{(i+1)n}y^i \tag{8}$$

by induction on m. For $m = 2n$, by (6), we have

$$\begin{aligned} s_{2n} - n =&(s_{2n-1} - n)\sigma_1 - (s_{2n-2} - n)\sigma_2 + \cdots + (-1)^n(s_{n+1} - n)\sigma_{n-1} + (s_n - n)((-1)^{n+1}\sigma_n + 1) \\ =&ny(C_{2n-1}^nC_n^1 - C_{2n-2}^nC_n^2 + \cdots + (-1)^nC_{n+1}^nC_n^{n-1} + (-1)^{n+1}\sigma_n + 1). \end{aligned}$$

By Lemma 6,

$$C_{2n-1}^nC_n^1 - C_{2n-2}^nC_n^2 + \cdots + (-1)^nC_{n+1}^nC_n^{n-1}{=}C_{2n}^n - C_n^0 + (-1)^nC_n^n.$$

Then,

$$s_{2n} - n = ny(C_{2n}^n - C_n^0 + (-1)^nC_n^n + (-1)^{n+1}\sigma_n + 1) = ny(C_{2n}^n + y) = ny\sum_{i=0}^{1} C_{2n}^{(i+1)n}y^i.$$

Suppose that $s_m - n = ny\sum\limits_{i=0}^{k-1} C_m^{(i+1)n}y^i$ is true for $2n \leqslant m \leqslant tn + r$, where $t \geqslant 2$ and $0 \leqslant r \leqslant n-1$. Note that when $(t-1)n \leqslant m < tn$, we have $C_m^{tn} = 0$. Then $\sum\limits_{i=0}^{t-2} C_{tn}^{(i+1)n}y^i = \sum\limits_{i=0}^{t-1} C_m^{(i+1)n}y^i$. Suppose that $m = tn + r + 1$. Then,

$$\begin{aligned} s_{tn+r+1} - n =&(s_{tn+r} - n)\sigma_1 - (s_{tn+r-1} - n)\sigma_2 + \cdots + (-1)^n(s_{tn+r-(n-2)} - n)\sigma_{n-1} \\ &+ (-1)^{n+1}(s_{tn+r-(n-1)} - n)\sigma_n + ny. \end{aligned}$$

When $0 \leqslant r \leqslant n-1$, we have $tn + 1 \leqslant tn + r + 1 \leqslant (t+1)n$. By induction,

$$\begin{aligned} s_{tn+r+1} - n =&ny\sum_{i=0}^{t-1}(\sum_{j=1}^{n-1}(-1)^{j+1}C_{tn+r-(j-1)}^{(i+1)n}C_n^j + (-1)^{n+1}C_{tn+r-(n-1)}^{(i+1)n}\sigma_n)y^i + ny \\ =&ny\sum_{i=0}^{t-1}(\sum_{j=1}^{n}(-1)^{j+1}C_{tn+r-(j-1)}^{(i+1)n}C_n^j + C_{tn+r-(n-1)}^{(i+1)n}y)y^i + ny. \end{aligned}$$

By Lemma 6,

$$\sum_{j=1}^{n}(-1)^{j+1}C_{tn+r-(j-1)}^{(i+1)n}C_n^j=C_{tn+r+1}^{(i+1)n}-C_{tn+r+1-n}^{in}.$$

Then,

$$\begin{aligned}s_{tn+r+1}-n=&ny\sum_{i=0}^{t-1}(C_{tn+r+1}^{(i+1)n}-C_{tn+r+1-n}^{in}+C_{tn+r-(n-1)}^{(i+1)n}y)y^i+ny\\=&ny\sum_{i=0}^{t-1}C_{tn+r+1}^{(i+1)n}y^i+nC_{tn+r-(n-1)}^{tn}y^{t+1}.\end{aligned}$$

• When $0\leqslant r<n-1$, that is $tn+1\leqslant tn+r+1<(t+1)n$, and $tn+r-(n-1)<tn$. We have $C_{tn+r-(n-1)}^{tn}=0$, thus $s_{tn+r+1}-n=ny\sum\limits_{i=0}^{t-1}C_{tn+r+1}^{(i+1)n}y^i$.

• When $r=n-1$, that is $tn+r+1=(t+1)n$, and $tn+r-(n-1)=tn$. We have $C_{tn+r-(n-1)}^{tn}=1$, thus $s_{tn+r+1}-n=ny\sum\limits_{i=0}^{t-1}C_{tn+r+1}^{(i+1)n}y^i+ny^{t+1}=ny\sum\limits_{i=0}^{t}C_{tn+r+1}^{(i+1)n}y^i$ and thus (8) is true.

However, by the last equality of (5), we have $s_m=n$. Suppose $m\geqslant 2n$, and thus $kn\leqslant m<(k+1)n$ where $k\geqslant 2$. Then by (8) we have $y=0$ or $\sum\limits_{i=0}^{k-1}C_m^{(i+1)n}y^i=0$. When $y=0$, we have $\sigma_n=1$. So $x_1,x_2,\cdots,x_n$ are the roots of $x^n-C_n^1x^{n-1}+C_n^2x^{n-2}-\cdots+(-1)^{n-1}C_n^1x+(-1)^n=(x-1)^n=0$. Therefore, the system of equations has solution $(1,1,\cdots,1)$. Let $f(y)=\sum\limits_{i=0}^{k-1}C_m^{(i+1)n}y^i$. Since $f(0)=C_m^n\neq 0$, it follows that the system of equations has solutions other than $(1,1,\cdots,1)$.

Therefore, the original system of equations has a unique solution $(1,1,\cdots,1)$ if and only if $n\leqslant m<2n$.

(ii) For the n-th primitive root ω of 1, we have

$$0=1-(\omega^n)^k=1-(\omega^k)^n=(1-\omega^k)[1+\omega^k+(\omega^2)^k+\cdots+(\omega^{n-1})^k],$$

where $k=1,2,\cdots,n-1$. Then,

$$\begin{cases}\omega+\omega^2+\cdots+\omega^n=0\\ \omega^2+(\omega^2)^2+\cdots+(\omega^2)^n=0\\ \qquad\vdots\\ \omega^{n-1}+(\omega^{n-1})^2+\cdots+(\omega^{n-1})^n=0.\end{cases}$$

Let $x_1=1+t\omega,x_2=1+t\omega^2,\cdots,x_n=1+t\omega^n$. Then we have

$$\begin{cases}x_1+x_2+\cdots+x_n=n\\ x_1^2+x_2^2+\cdots+x_n^2=n\\ \qquad\vdots\\ x_1^{n-1}+x_2^{n-1}+\cdots+x_n^{n-1}=n\end{cases}$$

and

$$
\begin{aligned}
&x_1^m + x_2^m + \cdots + x_n^m \\
=&(1+t\omega)^m + (1+t\omega^2)^m + \cdots + (1+t\omega^n)^m \\
=&\sum_{i=1}^{n} [C_m^0 + C_m^1(t\omega^i) + C_m^2(t\omega^i)^2 + \cdots + C_m^{kn}(t\omega^i)^{kn} + \cdots + C_m^m(t\omega^i)^m] \\
=&\sum_{i=1}^{n} C_m^0 + \sum_{i=1}^{n} C_m^1(t\omega^i) + \sum_{i=1}^{n} C_m^2(t\omega^i)^2 + \cdots + \sum_{i=1}^{n} C_m^{kn}(t\omega^i)^{kn} + \cdots + \sum_{i=1}^{n} C_m^m(t\omega^i)^m \\
=&n + nt^n \sum_{i=0}^{k-1} C_m^{(i+1)n}(t^n)^i \\
=&n.
\end{aligned}
$$

Thus $(1+t\omega, 1+t\omega^2, \cdots, 1+t\omega^n)$ is a solution of the system of equations.

Conversely, let $(x_1, x_2, \cdots, x_n)$ be any solution of the system of equations. We have

$$
\begin{aligned}
f(x) = \prod_{i=1}^{n}(x-x_i) &= x^n - \sigma_1 x^{n-1} + \sigma_2 x^{n-2} - \cdots + (-1)^n \sigma_n \\
&= x^n - C_n^1 x^{n-1} + C_n^2 x^{n-2} - \cdots + (-1)^{n-1} C_n^{n-1} x + (-1)^n - y \\
&= (x-1)^n - y.
\end{aligned}
$$

By (i) we have that y satisfies $g(y) = y \sum_{i=0}^{k-1} C_m^{(i+1)n} y^i$. We may as well let $y = t^n$, $t \in \mathbb{C}$. Then,

$$f(x) = (x-1)^n - t^n = (x-1-t\omega)(x-1-t\omega^2)\cdots(x-1-t\omega^n).$$

Thus the root of polynomial $f(x)$ is $(1+t\omega, 1+t\omega^2, \cdots, 1+t\omega^n)$, where t is a root of $f(x) = x^n \sum_{i=0}^{k-1} C_m^{(i+1)n}(x^n)^i$. This completes the proof. □

3 Proof of Theorem 2

Proof of Theorem 2. By (1) and induction on i, it is easily obtained that $\sigma_1 = 1$, $\sigma_i = 0, 2 \leqslant i \leqslant n-1$. Let $y = (-1)^{n+1}\sigma_n$. Then by (1) and (2) we have

$$
\begin{cases}
s_n - 1 = ny \\
s_m - s_{m-1} = s_{m-n} y, \quad m > n.
\end{cases}
\tag{9}
$$

For $n \leqslant m < 2n$, we prove

$$s_m = 1 + my \tag{10}$$

by induction on m. When $m = n$, by (9), we have $s_n = 1 + ny$, the result is true. Assume that $s_m = 1 + my$ holds for $m = k, n \leqslant k < 2n-1$, that is $s_k = 1 + ky$. Now suppose that $m = k+1$. By (9), $s_{k+1} = s_k + s_{k+1-n} y$, and $k+1-n < n$, so $s_{k+1-n} = 1$. By induction,

$$s_{k+1} = (1+ky) + y = 1 + (k+1)y.$$

Thus, (10) holds for $n \leqslant m < 2n$. However, $s_m = 1$, then $my = 0$, so $y = 0$ and thus $\sigma_n = 0$. Therefore, $x_1, x_2, \cdots, x_n$ are the roots of $x^n - x^{n-1} = x^{n-1}(x-1) = 0$, and thus the system of equations has a unique solution $(1, 0, \cdots, 0)$. In fact, note that we have $s_m - 1 = my = \frac{m}{n}(s_n - 1)$. When $n \leqslant m < 2n$, the system of equations is equivalent to

$$\begin{cases} x_1 + x_2 + \cdots + x_n = 1 \\ x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \\ \qquad \vdots \\ x_1^n + x_2^n + \cdots + x_n^n = 1. \end{cases}$$

When $2n \leqslant m < 3n$, we prove

$$s_m - 1 = my(\frac{1}{2}C^1_{m-2n+1}y + 1) \tag{11}$$

by induction on m. When $m = 2n$, we have $(s_{2n} - 1) = (s_{2n-1} - 1) + s_n y$ by (9). Together with (10), we obtain $(s_{2n} - 1) = ny(y + 2)$. The result is true. Assume that $s_m - 1 = my(\frac{1}{2}C^1_{m-2n+1}y + 1)$ holds for $m = k, 2n \leqslant k < 3n-1$, that is, $s_k - 1 = ky(\frac{1}{2}C^1_{k-2n+1}y+1)$. Now we assume that $m = k + 1$. By (9), we have $s_{k+1} - 1 = (s_k - 1) + s_{k+1-n}y$, and $n < k+1-n < 2n$. Then by (10), we have $s_{k+1-n} = 1 + (k+1-n)y$. Thus by induction, we have $s_{k+1} - 1 = (k + 1)y(\frac{1}{2}C^1_{k+2-2n}y + 1)$ and (11) is true. Since $s_m = 1$, we have $my(\frac{1}{2}C^1_{m-2n+1}y + 1) = 0$ by (11). Then $y = 0$ or $\frac{1}{2}C^1_{m-2n+1}y + 1 = 0$.

If $y = 0$, then $\sigma_n = 0$. So $x_1, x_2, \cdots, x_n$ are the roots of $f(x) = x^n - x^{n-1} = x^{n-1}(x-1)$. Therefore, the system of equations has a solution $(1, 0, \cdots, 0)$.

If $\frac{1}{2}C^1_{m-2n+1}y + 1 = 0$, then we obtain $\sigma_n = (-1)^n \frac{2}{m-2n+1} \neq 0$. So the system of equations has solutions other than $(1, 0, \cdots, 0)$, which are the roots of $f(x) = x^n - x^{n-1} + \frac{2}{m-2n+1}$.

Suppose $m \geqslant 3n$, and $kn \leqslant m < (k+1)n, k = 3, 4, \cdots$. Then we prove

$$s_m - 1 = my \sum_{i=0}^{k-1} \frac{1}{k-i} C^{k-i-1}_{m-(k-i)n+k-i-1} y^{k-i-1} \tag{12}$$

by induction on m. If $m = 3n$, then we have $s_{3n} - 1 = (s_{3n-1} - 1) + s_{2n}y$ by (9). Together with (11), $s_{3n} - 1 = 3ny(\frac{1}{3}y^n + \frac{1}{2}C^1_{n+1}y + 1)$. The result is true. Suppose that $s_m - 1 = my \sum_{i=0}^{k-1} \frac{1}{k-i} C^{k-i-1}_{m-(k-i)n+k-i-1} y^{k-i-1}$ holds for $3n \leqslant m \leqslant tn + r$, where $t = 3, 4, \cdots; r = 0, 1, \cdots, n-1$. Let $m = tn+r+1$. Then $s_{tn+r+1}-1 = (s_{tn+r}-1)+s_{(t-1)n+r+1}y$. Since $0 \leqslant r \leqslant n-1, t \geqslant 3$, we see that $2n+1 \leqslant (t-1)n+r+1 \leqslant tn \leqslant tn+r$. By (11), we have that (12) is true when $2n + 1 \leqslant m \leqslant 3n$. So s_{tn+r} and $s_{(t-1)n+r+1}$ can be expressed by induction.

- When $(t-1)n \leqslant (t-1)n + r + 1 < tn$, that is, $tn \leqslant tn + r + 1 < (t+1)n$, we have

$$s_{tn+r} - 1 = (tn + r)y \sum_{i=0}^{t-1} \frac{1}{t-i} C^{t-i-1}_{tn+r-(t-i)n+t-i-1} y^{t-i-1},$$

$$s_{(t-1)n+r+1}-1=((t-1)n+r+1)y\sum_{i=0}^{t-2}\frac{1}{t-1-i}C^{t-1-i-1}_{(t-1)n+r+1-(t-1-i)n+t-1-i-1}y^{t-1-i-1}.$$

Then

$$\begin{aligned}
&s_{tn+r+1}-1=(s_{tn+r}-1)+s_{(t-1)n+r+1}y\\
=&\sum_{i=0}^{t-2}(\frac{tn+r}{t-i}C^{t-i-1}_{i(n-1)+r+t-1}+\frac{(t-1)n+r+1}{t-1-i}C^{t-i-2}_{i(n-1)+r+t-1})y^{t-i}+(tn+r+1)y\\
=&\sum_{i=0}^{t-2}(tn+r+1)\frac{1}{t-i}C^{t-i-1}_{tn+r+1-(t-i)n+t-i-1}y^{t-i}+(tn+r+1)y\\
=&(tn+r+1)y\sum_{i=0}^{t-1}\frac{1}{t-i}C^{t-i-1}_{tn+r+1-(t-i)n+t-i-1}y^{t-i-1}.
\end{aligned}$$

So (12) holds.

• When $(t-1)n+r+1=tn$, that is, $tn+r+1=(t+1)n$, we have

$$s_{(t+1)n}-1=(s_{(t+1)n-1}-1)+s_{tn}y=((t+1)n-1)y\sum_{i=0}^{t-1}\frac{1}{t-i}C^{t-i-1}_{(t+1)n-1-(t-i)n+t-i-1}y^{t-i-1},$$

$$s_{tn}-1=tny\sum_{i=0}^{t-1}\frac{1}{t-i}C^{t-i-1}_{tn-(t-i)n+t-i-1}y^{t-i-1}.$$

Then

$$\begin{aligned}
&s_{(t+1)n}-1=(s_{(t+1)n-1}-1)\sigma_1+s_{tn}y\\
=&((t+1)n-1)\sum_{i=0}^{t-1}\frac{1}{t-i}C^{t-i-1}_{t+n-2+i(n-1)}y^{t-i}+tn\sum_{i=0}^{t-1}\frac{1}{t-i}C^{t-i-1}_{t-1+i(n-1)}y^{t-i+1}+y\\
=&((t+1)n-1)\sum_{i=1}^{t}\frac{1}{t-i+1}C^{t-i}_{t+n-2+(i-1)(n-1)}y^{t-i+1}+tn\sum_{i=0}^{t-1}\frac{1}{t-i}C^{t-i-1}_{t-1+i(n-1)}y^{t-i+1}+y\\
=&\sum_{i=1}^{t-1}(\frac{(t+1)n-1}{t-i+1}C^{t-i}_{t-1+i(n-1)}+\frac{tn}{t-i}C^{t-i-1}_{t-1+i(n-1)})y^{t-i+1}+[(t+1)n-1]y+ny^{t+1}+y\\
=&\sum_{i=1}^{t-1}\frac{(t+1)n}{t+1-i}C^{t+1-i-1}_{(t+1)n-(t+1-i)n+(t+1-i-1)}y^{t-i+1}+(t+1)ny+ny^{t+1}\\
=&(t+1)ny\sum_{i=0}^{t}\frac{1}{t+1-i}C^{t+1-i-1}_{(t+1)n-(t+1-i)n+(t+1-i-1)}y^{t-i}.
\end{aligned}$$

Therefore, the conclusion is also true.

To summarize, when $kn\leqslant m<(k+1)n, k=2,3,\cdots,$ (12) is true.

Since $s_m = 1$, it follows that $y = 0$ or $\sum_{i=0}^{k-1} \frac{1}{k-i} C_{m-(k-i)n+k-i-1}^{k-i-1} y^{k-i-1} = 0$. If $y = 0$, then $\sigma_n = 0$. So $x_1, x_2, \cdots, x_n$ are the roots of $f(x) = x^n - x^{n-1} = x^{n-1}(x-1)$. Therefore, the system of equations has solution $(1, 0, \cdots, 0)$. Let $f(y) = \sum_{i=0}^{k-1} \frac{1}{k-i} C_{m-(k-i)n+k-i-1}^{k-i-1} y^{k-i-1}$. Then $f(0) = 1 \neq 0$. So the system of equations has solutions other than $(1, 0, \cdots, 0)$, satisfying $f(x) = x^n - x^{n-1} - y$, where y is a root of $f(y) = \sum_{i=0}^{k-1} \frac{1}{k-i} C_{m-(k-i)n+k-i-1}^{k-i-1} y^{k-i-1}$.

Therefore, the original system of equations has a unique solution $(1, 0, \cdots, 0)$ if and only if $n \leqslant m < 2n$. □

4 Proof of Theorems 3 and 4

Proof of Theorem 3. By (1) and induction, it is easily obtained that

$$\sigma_1 = \sigma_2 = \cdots = \sigma_{n-1} = 0. \tag{13}$$

Assume that $kn \leqslant m < (k+1)n, k = 1, 2, \cdots$. By (13), we have

$$\begin{cases} s_n + (-1)^n n\sigma_n = 0 \\ s_m + (-1)^n s_{m-n}\sigma_n = 0, m > n. \end{cases}$$

If $m = kn$, we have

$$s_m = [(-1)^{n+1}]^{k-1} s_n \sigma_n^{k-1} = [(-1)^{n+1}]^k s_n \sigma_n^k = [(-1)^{n+1}]^{k+1} n\sigma_n^{k+1}.$$

Since $s_m = 0$, we see that $\sigma_n^k = 0$. So $\sigma_n = 0$. Therefore, $x_1, x_2, \cdots, x_n$ are the roots of $x^n = 0$. So the system of equations has a unique solution $(0, 0, \cdots, 0)$.

If $kn < m < (k+1)n$, then $s_m = [(-1)^{n+1}]^k s_{m-kn}\sigma_n^k$. Since $0 < m - kn < n$, we have $s_{m-kn} = 0$. As $s = 0$, σ_n can take any complex number. We may as well assume that $\sigma_n = (-1)^{n+1}a^n$, where a takes any complex number. Thus $x_1, x_2, \cdots, x_n$ are the roots of $x^n + (-1)^n\sigma_n = x^n - a^n = (x - a\omega)(x - a\omega^2) \cdots (x - a\omega^n) = 0$. So the system of equations has the solution $(a\omega, a\omega^2, \cdots, a\omega^n)$. □

While $a = 0$, naturally, we consider another set of special values for $\{k_1, k_2, \cdots, k_n\}$, that is, $\{k_1, k_2, \cdots, k_n\}$ take consecutive natural numbers.

Proof of Theorem 4. It is obvious that $m = 1$ is true by Theorem 3, so we only need to discuss the case when $m > 1$.

Assume that $1 < m \leqslant n$. By (1) and (2),

$$\begin{cases} s_{m+n-1} - s_{m+n-2}\sigma_1 + s_{m+n-3}\sigma_2 - \cdots + (-1)^{n-1} s_m\sigma_{n-1} + (-1)^n s_{m-1}\sigma_n = 0 \\ s_{m+n-2} - s_{m+n-3}\sigma_1 + s_{m+n-4}\sigma_2 - \cdots + (-1)^{n-1} s_{m-1}\sigma_{n-1} + (-1)^n s_{m-2}\sigma_n = 0 \\ \vdots \\ s_m - s_{m-1}\sigma_1 + s_{m-2}\sigma_2 - \cdots + (-1)^{m-1} s_1\sigma_{m-1} + (-1)^m m\sigma_m = 0. \end{cases}$$

Therefore, we obtain

$$\begin{cases} s_{m-1}\sigma_n = 0 \\ s_{m-2}\sigma_n - s_{m-1}\sigma_{n-1} = 0 \\ s_{m-3}\sigma_n - s_{m-2}\sigma_{n-1} + s_{m-1}\sigma_{n-2} = 0 \\ \vdots \\ m\sigma_n - s_m\sigma_{m-1} + \cdots + (-1)^{m-2}s_{m-2}\sigma_2 + (-1)^{m-1}s_{m-1}\sigma_1 = 0. \end{cases}$$

Thus,

$$\begin{cases} s_{m-1} = s_{m-2} = \cdots = s_1 = 0, \sigma_n = \sigma_{n-1} = \cdots = \sigma_m = 0 \\ s_{m-1} = s_{m-2} = \cdots = s_{m-i} = 0, \sigma_n = \sigma_{n-1} = \cdots = \sigma_{i+1} = 0, 1 \leqslant i \leqslant m-2 \\ \sigma_n = \sigma_{n-1} = \cdots = \sigma_1 = 0. \end{cases} \tag{14}$$

Consider the solution at i. Since

$$s_{m-1}+\sum_{j=1}^{i-1}(-1)^j s_{m-1-j}\sigma_j+(-1)^i s_{m-1-i}\sigma_i+\sum_{j=i+1}^{m-2}(-1)^j s_{m-1-j}\sigma_j+(-1)^{m-1}(m-1)\sigma_{m-1} = 0,$$

It follows that $s_{m-1-i}\sigma_i = 0$. Thus $s_{m-1-i} = 0$ or $\sigma_i = 0$.

If $s_{m-1-i} = 0$, it comes down to the solution at $i+1$. While $\sigma_i = 0$, it comes down to the solution at $i-1$. Thus (14) can be reduced to

$$\begin{cases} s_{m-1} = s_{m-2} = \cdots = s_1 = 0, \sigma_n = \sigma_{n-1} = \cdots = \sigma_m = 0 \\ \sigma_n = \sigma_{n-1} = \cdots = \sigma_1 = 0. \end{cases}$$

However, by $s_{m-1} = s_{m-2} = \cdots = s_1 = 0$, we have $\sigma_1 = \sigma_2 = \cdots = \sigma_{m\text{-}1} = 0$. So $\sigma_n = \sigma_{n-1} = \cdots = \sigma_1 = 0$. Thus the system of equations has a unique solution.

Suppose that $kn < m \leqslant (k+1)n$, for $k = 1, 2, \cdots$. By (2),

$$\begin{cases} s_{m+n-1} - s_{m+n-2}\sigma_1 + s_{m+n-3}\sigma_2 - \cdots + (-1)^{n-1}s_m\sigma_{n-1} + (-1)^n s_{m-1}\sigma_n = 0 \\ s_{m+n-2} - s_{m+n-3}\sigma_1 + s_{m+n-4}\sigma_2 - \cdots + (-1)^{n-1}s_{m-1}\sigma_{n-1} + (-1)^n s_{m-2}\sigma_n = 0 \\ \vdots \\ s_m - s_{m-1}\sigma_1 + s_{m-2}\sigma_2 - \cdots + (-1)^{n-1}s_{m-(n-1)}\sigma_{n-1} + (-1)^n s_{m-n}\sigma_n = 0. \end{cases}$$

Therefore, we obtain

$$\begin{cases} s_{m-1}\sigma_n = 0 \\ s_{m-2}\sigma_n - s_{m-1}\sigma_{n-1} = 0 \\ s_{m-3}\sigma_n - s_{m-2}\sigma_{n-1} + s_{m-1}\sigma_{n-2} = 0 \\ \vdots \\ s_{m-n}\sigma_n - s_{m-(n-1)}\sigma_{n-1} + \cdots + (-1)^{n-2}s_{m-2}\sigma_2 + (-1)^{n-1}s_{m-1}\sigma_1 = 0. \end{cases}$$

Then,

$$\begin{cases} s_{m-1} = s_{m-2} = \cdots = s_{m-n} = 0 \\ s_{m-1} = s_{m-2} = \cdots = s_{m-i} = 0, \sigma_n = \sigma_{n-1} = \cdots = \sigma_{i+1} = 0, 1 \leqslant i \leqslant n-1. \end{cases} \tag{15}$$

If $m = n+1$, we have

$$s_n + \sum_{j=1}^{i-1} (-1)^j s_{n-j}\sigma_j + (-1)^i s_{n-i}\sigma_i + \sum_{j=i+1}^{n-1} (-1)^j s_{n-j}\sigma_j + (-1)^n n\sigma_n = 0.$$

So we have $s_{n-i}\sigma_i = 0$. Hence $s_{n-i} = 0$ or $\sigma_i = 0$.

If $s_{n-i} = 0$, then it comes down to the solution at $i+1$. While $\sigma_i = 0$, it comes down to the solution at $i-1$. But note that when $i = 1$, we have $\sigma_1 = \sigma_2 = \cdots = \sigma_n = 0$. In this case the solution is unique. So we just have to consider the first component of (15), that is, $s_n = s_{n-1} = \cdots = s_1 = 0$, it is obvious that the solution is unique.

If $kn + 1 < m \leqslant (k+1)n$, we obtain

$$s_{m-1} + \sum_{j=1}^{i-1} (-1)^j s_{m-1-j}\sigma_j + (-1)^i s_{m-1-i}\sigma_i + \sum_{j=i+1}^{n} (-1)^j s_{m-1-j}\sigma_j = 0.$$

So we have $s_{m-1-i}\sigma_i = 0$. Thus $s_{m-1-i} = 0$ or $\sigma_i = 0$.

If $s_{m-1-i} = 0$, then it comes down to the solution at $i+1$. While $\sigma_i = 0$, it comes down to the solution at $i-1$. Note that when $i = 1$, we have $\sigma_1 = \sigma_2 = \cdots = \sigma_n = 0$. In this case the solution is unique. So we just have to consider the first component of (15), that is, $s_{m-1} = s_{m-2} = \cdots = s_{m-n} = 0$. By recurrence, we obtain $s_{m-1-(k-1)n} = s_{m-2-(k-1)n} = \cdots = s_{m-kn} = 0$. Now we have $m - kn \leqslant n$, reducing to the case $1 < m \leqslant n$. So the system of equations has a unique solution. □

However, when $a \in \{1, n\}$ and $\{k_1, k_2, \cdots, k_n\}$ take consecutive natural numbers, the following example indicates that the solution is not necessarily unique.

Example 7. The solutions of the following system of equations $\begin{cases} x_1^2 + x_2^2 = 1 \\ x_1^3 + x_2^3 = 1 \end{cases}$ and $\begin{cases} x_1^2 + x_2^2 = 2 \\ x_1^3 + x_2^3 = 2 \end{cases}$ are not unique.

Proof. We have

$$\begin{cases} x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = \sigma_1^2 - 2\sigma_2 \\ x_1^3 + x_2^3 = (x_1 + x_2)^3 - 3x_1x_2(x_1 + x_2) = \sigma_1^3 - 3\sigma_1\sigma_2 \end{cases}$$

Assume that $\begin{cases} \sigma_1^2 - 2\sigma_2 = 1 \\ \sigma_1^3 - 3\sigma_1\sigma_2 = 1 \end{cases}$. We obtain $(\sigma_1 - 1)^2(\sigma_1 + 2) = 0$. So $\begin{cases} x_1^2 + x_2^2 = 1 \\ x_1^3 + x_2^3 = 1 \end{cases}$ has solutions other than $(1, 0)$.

Suppose that $\begin{cases} \sigma_1^2 - 2\sigma_2 = 2 \\ \sigma_1^3 - 3\sigma_1\sigma_2 = 2 \end{cases}$. We obtain $(\sigma_1 - 2)(\sigma_1^2 - 2\sigma_1 - 2) = 0$. So the equations $\begin{cases} x_1^2 + x_2^2 = 2 \\ x_1^3 + x_2^3 = 2 \end{cases}$ has solutions other than $(1, 1)$. □

5 Some Examples and Proof of Theorem 5

The following examples indicate that it may be very complicated to solving all arrays $\{k_1, k_2, \cdots, k_n\}$ that make (4) have a unique solution, whenever $a = 0, 1, n$.

Example 8. Let $n, r \in \mathbb{N}^+$, $n \geqslant 4, 1 \leqslant r \leqslant n-3$. Then the system of equations of sum of equal powers

$$\begin{cases} x_1 + x_2 + \cdots + x_n = n \\ x_1^2 + x_2^2 + \cdots + x_n^2 = n \\ \qquad\qquad \vdots \\ x_1^{n-2} + x_2^{n-2} + \cdots + x_n^{n-2} = n \\ x_1^n + x_2^n + \cdots + x_n^n = n \\ x_1^{n+r} + x_2^{n+r} + \cdots + x_n^{n+r} = n \end{cases} \tag{16}$$

has a unique solution $(1, 1, \cdots, 1)$.

Proof. By (1), we have

$$(s_n - n) + \sum_{i=1}^{n-1} (-1)^i (s_{n-i} - n) \sigma_i + (-1)^{n-2} n(n-1) + (-1)^{n-1} n\sigma_{n-1} + (-1)^n n\sigma_n = 0. \tag{17}$$

Assume $1 \leqslant t \leqslant r$. By (2), we obtain

$$(s_{n+t} - n) + \sum_{i=1}^{n} (-1)^i (s_{n+t-i} - n) \sigma_i + (-1)^{n-2} n(n-1) + (-1)^{n-1} n\sigma_{n-1} + (-1)^n n\sigma_n = 0. \tag{18}$$

Since $s_1 - n = s_2 - n = \cdots = s_{n-2} - n = s_n - n = s_{n+r} - n = 0$, it follows from (17), $(s_{n-1} - n)\sigma_1 = (-1)^{n-2} n(n-1) + (-1)^{n-1} n\sigma_{n-1} + (-1)^n n\sigma_n$. Substitute it into (18), we obtain:

$$\begin{cases} \sum\limits_{i=1}^{r-1} (-1)^i (s_{n+r-i} - n)\sigma_i + ((-1)^{r+1}\sigma_{r+1} + \sigma_1)(s_{n-1} - n) = 0 \\ (s_{n+r-1} - n) + \sum\limits_{i=1}^{r-2} (-1)^i (s_{n+r-1-i} - n)\sigma_i + ((-1)^r \sigma_r + \sigma_1)(s_{n-1} - n) = 0 \\ (s_{n+r-2} - n) + \sum\limits_{i=1}^{r-3} (-1)^i (s_{n+r-2-i} - n)\sigma_i + ((-1)^{r-1}\sigma_{r-1} + \sigma_1)(s_{n-1} - n) = 0 \\ \vdots \\ (s_{n+2} - n) - \sigma_1(s_{n+1} - n) + (-\sigma_3 + \sigma_1)(s_{n-1} - n) = 0 \\ (s_{n+1} - n) + (\sigma_2 + \sigma_1)(s_{n-1} - n) = 0. \end{cases}$$

Let

$$\begin{cases} z_{n+r-1} = s_{n+r-1} - n \\ z_{n+r-2} = s_{n+r-2} - n \\ \vdots \\ z_{n+1} = s_{n+1} - n \\ z_{n-1} = s_{n-1} - n. \end{cases}$$

The system of equations are transformed into

$$\begin{cases} -C_n^1 z_{n+r-1}+C_n^2 z_{n+r-2}-\cdots+(-1)^{r-1}C_n^{r-1}z_{n+1}+((-1)^{r+1}C_n^{r+1}+C_n^1)z_{n-1}{=}0\\ z_{n+r-1}-C_n^1 z_{n+r-2}+\cdots+(-1)^{r-2}C_n^{r-2}z_{n+1}+((-1)^{r}C_n^{r}+C_n^1)z_{n-1}{=}0\\ z_{n+r-2}-C_n^1 z_{n+r-3}+\cdots+(-1)^{r-3}C_n^{r-3}z_{n+1}+((-1)^{r-1}C_n^{r-1}+C_n^1)z_{n-1}{=}0\\ \vdots\\ z_{n+2}-C_n^1 z_{n+1}+(-C_n^3+C_n^1)z_{n-1}{=}0\\ z_{n+1}+(C_n^2+C_n^1)z_{n-1}{=}0. \end{cases}$$

Consider the coefficient matrix

$$A=\begin{pmatrix} -C_n^1 & C_n^2 & -C_n^3 & \cdots & (-1)^{r-1}C_n^{r-1} & (-1)^{r+1}C_n^{r+1}+C_n^1\\ 1 & -C_n^1 & C_n^1 & \cdots & (-1)^{r-2}C_n^{r-2} & (-1)^{r}C_n^{r}+C_n^1\\ 0 & 1 & -C_n^1 & \cdots & (-1)^{r-3}C_n^{r-3} & (-1)^{r-1}C_n^{r-1}+C_n^1\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & -C_n^1 & -C_n^3+C_n^1\\ 0 & 0 & 0 & \cdots & 1 & C_n^2+C_n^1. \end{pmatrix}$$

We obtain $|A|=(-1)^{r+1}rC_{n+r}^{r+1}\neq 0$. So the system of equations only has zero solution. Thus $s_{n-1}=n$. By Theorem 1, we obtain that the original system of equations has a unique solution $(1,1,\cdots,1)$. □

Example 9. Let $n,t\in\mathbb{N}^+$, $n=2t+1, n\geqslant 3, 1\leqslant t\leqslant n-1$. Then the system of equations of sum of equal powers

$$\begin{cases} x_1+x_2+\cdots+x_n=1\\ x_1^2+x_2^2+\cdots+x_n^2=1\\ \qquad\vdots\\ x_1^t+x_2^t+\cdots+x_n^t=1\\ x_1^{t+2}+x_2^{t+2}+\cdots+x_n^{t+2}=1\\ x_1^{t+3}+x_2^{t+3}+\cdots+x_n^{t+3}=1\\ \qquad\vdots\\ x_1^{n+1}+x_2^{n+1}+\cdots+x_n^{n+1}=1 \end{cases} \tag{19}$$

has a unique solution (ignoring order) $(1,0,\cdots,0)$.

Proof. Note that we have $s_1=s_2=\cdots=s_t=s_{t+2}=s_{t+3}=\cdots=s_{n+1}=1$, and $\sigma_1=1,\sigma_2=\sigma_3=\cdots=\sigma_t=0$, substituting to (1) and (2), we obtain

$$\begin{cases} s_{t+1}-1+(-1)^{t+1}(t+1)\sigma_{t+1}=0\\ s_{t+2}-s_{t+1}+(-1)^{t+1}\sigma_{t+1}+(-1)^{t+2}(t+2)\sigma_{t+2}=0\\ (-1)^{t+1}\sigma_{t+1}+(-1)^{t+2}\sigma_{t+2}+(-1)^{t+3}(t+3)\sigma_{t+3}=0\\ \qquad\vdots\\ (-1)^{t+1}\sigma_{t+1}+(-1)^{t+2}\sigma_{t+2}+\cdots+(-1)^{n-1}(n-1)\sigma_{n-1}=0\\ (-1)^{t+1}\sigma_{t+1}+(-1)^{t+2}\sigma_{t+2}+\cdots+(-1)^{n}n\sigma_{n}=0\\ (-1)^{t+1}s_{t+1}\sigma_{t+1}+(-1)^{t+2}\sigma_{t+2}+\cdots+(-1)^{n}\sigma_{n}=0. \end{cases} \tag{20}$$

By the first two equations of (20), we have $\sigma_{t+1} = \sigma_{t+2}$. Then substitute it into from the third to the $(t+1)$-th equation of (20) in turn, we obtain $\sigma_{t+3} = \sigma_{t+4} = \cdots = \sigma_n = 0$. Substituting it to the $(t+2)$-th equation of (20), we have $s_{t+1}\sigma_{t+1} - \sigma_{t+2} = 0$. Then $(s_{t+1}-1)\sigma_{t+1} = 0$, $s_{t+1} = 1$ or $\sigma_{t+1} = 0$. Together with the first equation of (20), we obtain that $s_{t+1} = 1$ is equivalent to $\sigma_{t+1} = 0$. Then $\sigma_{t+2} = 0$. Thus, σ_{n-1}=0. So the system of equations has a unique solution (ignoring order) $(1, 0, \cdots, 0)$. $\square$

When $a = 0$, the array $\{k_1, k_2, \cdots, k_n\}$ in Theorem 5 satisfies the unique solution condition.

Proof of Theorem 5. By (1) and (2), we have

$$\begin{cases} s_l - \sigma_1 s_{l-1} + \sigma_2 s_{l-2} - \cdots + (-1)^{l-1}\sigma_{l-1}s_1 + (-1)^l l\sigma_l = 0, 1 \leqslant l \leqslant n; \\ s_l - \sigma_1 s_{l-1} + \sigma_2 s_{l-2} - \cdots + (-1)^n \sigma_n s_{l-n} = 0, l > n. \end{cases} \tag{21}$$

Since $s_1 = s_2 = \cdots = s_k = s_{k+2} = \cdots = s_n = s_m = 0$, we have $\sigma_1 = \sigma_2 = \cdots = \sigma_k = 0$. Substituting it into other equations of (21) in turn, we obtain

$$\begin{cases} \sigma_{u(k+1)+u_1} = 0, u = 1, 2, \cdots, p-1, u_1 = 1, 2, \cdots, k \\ \sigma_{u(k+1)} = \dfrac{1}{u!}\sigma_{k+1}^u, u = 1, 2, \cdots, p \\ \sigma_{p(k+1)+1} = \sigma_{p(k+1)+2} = \cdots = \sigma_{p(k+1)+r_1} = 0 \end{cases} \tag{22}$$

and

$$\begin{cases} s_{p(k+1)+r_1+1} = s_{p(k+1)+r_1+2} = \cdots = s_{p(k+1)+k} = 0 \\ s_{v(k+1)+v_1} = 0, v = p+1, \cdots, q-1, v_1 = 1, 2, \cdots, k \\ s_{v(k+1)} = (-1)^{v(k+2)-p-1}\dfrac{k+1}{(v-p-1)!p!}\sigma_{k+1}^v, v = p+1, \cdots, q \\ s_{q(k+1)+1} = s_{q(k+1)+2} = \cdots = s_{q(k+1)+r_2} = 0. \end{cases} \tag{23}$$

(i) When $r_2 = 0$, that is, $(k+1)|m$, we have $s_{q(k+1)} = (-1)^{q(k+2)-p-1}\frac{k+1}{(q-p-1)!p!}\sigma_{k+1}^q = 0$. Then $\sigma_{k+1} = 0$, thus $\sigma_{u(k+1)} = \frac{1}{u!}\sigma_{k+1}^u, u = 1, 2, \cdots, p$. Together with (22), we have $\sigma_i = 0, i = 1, 2, \cdots, n$. So the system of equations only has zero solution;

(ii) When $r_2 \neq 0$, by (23), $s_m = s_{q(k+1)+r_2} = 0$. So σ_{k+1} can take any complex number. We may as well let $(-1)^{k+1}\sigma_{k+1} = c$. Then together with (22), we obtain the solution is the root of $x^{n-p(k+1)}(x^{p(k+1)} + cx^{(p-1)(k+1)} + \frac{c^2}{2!}x^{(p-2)(k+1)} + \cdots + \frac{c^{(p-1)}}{(p-1)!}x^{k+1} + \frac{c^p}{p!}) = 0$, where c takes any complex number. $\square$

6 Conclusion

Systems of equations of algebraic sum of equal powers have wide applications in many areas such as communications, robotics, chemistry and mechanics, see [9]. It is also closely related to computer algebra and mathematical mechanization, see [10]. In [11, 12, 13], the authors use different methods to generalize (3) to the following systems of equations of algebraic sum of equal powers:

$$\begin{cases} x_1 + x_2 + \cdots + x_k - x_{k+1} - x_{k+2} - \cdots - x_n = p_1 \\ x_1^2 + x_2^2 + \cdots + x_k^2 - x_{k+1}^2 - x_{k+2}^2 - \cdots - x_n^2 = p_2 \\ \qquad\qquad \vdots \\ x_1^{n-1} + x_2^{n-1} + \cdots + x_k^{n-1} - x_{k+1}^{n-1} - x_{k+2}^{n-1} - \cdots - x_n^{n-1} = p_{n-1} \\ x_1^n + x_2^n + \cdots + x_k^n - x_{k+1}^n - x_{k+2}^n - \cdots - x_n^n = p_n, \end{cases}$$

where $p_i \in \mathbb{C}, i = 1, 2, \cdots, n$.

The systems of equations studied in this paper can be generalized to the corresponding systems of equations of algebraic sum of equal powers:

$$\begin{cases} x_1 + x_2 + \cdots + x_k - x_{k+1} - x_{k+2} - \cdots - x_n = a \\ x_1^2 + x_2^2 + \cdots + x_k^2 - x_{k+1}^2 - x_{k+2}^2 - \cdots - x_n^2 = a \\ \qquad\qquad \vdots \\ x_1^{n-1} + x_2^{n-1} + \cdots + x_k^{n-1} - x_{k+1}^{n-1} - x_{k+2}^{n-1} - \cdots - x_n^{n-1} = a \\ x_1^m + x_2^m + \cdots + x_k^m - x_{k+1}^m - x_{k+2}^m - \cdots - x_n^m = a, m > n. \end{cases}$$

How to solve the above systems of equations and an even more general situation

$$\begin{cases} x_1^{k_1} + x_2^{k_1} + \cdots + x_k^{k_1} - x_{k+1}^{k_1} - x_{k+2}^{k_1} - \cdots - x_n^{k_1} = a \\ x_1^{k_2} + x_2^{k_2} + \cdots + x_k^{k_2} - x_{k+1}^{k_2} - x_{k+2}^{k_2} - \cdots - x_n^{k_2} = a \\ \qquad\qquad \vdots \\ x_1^{k_{n-1}} + x_2^{k_{n-1}} + \cdots + x_k^{k_{n-1}} - x_{k+1}^{k_{n-1}} - x_{k+2}^{k_{n-1}} - \cdots - x_n^{k_{n-1}} = a \\ x_1^{k_n} + x_2^{k_n} + \cdots + x_k^{k_n} - x_{k+1}^{k_n} - x_{k+2}^{k_n} - \cdots - x_n^{k_n} = a \end{cases}$$

is worth to study in the future.

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