Some aspects of spectral graph theory

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Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, where $n = |V(G)|$ is the order and $m = |E(G)|$ is the size of G.

$d_G(u)$: the degree of vertex u in G.

Introduction

$A(G)$: the $(0,1)$ adjacency matrix of G.

$L(G)$: the Laplacian matrix of G.

$Q(G)$: the signless Laplacian matrix of G.

$D(G)$: the distance matrix of (connected) G.

$\rho_1(G) \geq \cdots \geq \rho_n(G) = \rho_{\min}(G)$: the (adjacency) eigenvalues of G.

$\lambda_1(G) \geq \cdots \geq \lambda_n(G)$: the Laplacian eigenvalues of G.

$\mu_1(G) \geq \cdots \geq \mu_n(G)$: the signless Laplacian eigenvalues of G.

$\gamma_1(G) \geq \cdots \geq \gamma_n(G)$: the distance eigenvalues of G.

(Adjacency) eigenvalues

A classical result (A.J. Hoffman, 1970) is: for a nonempty graph G on n vertices,

$$\chi(G) \geq 1 + \frac{\rho_1}{-\rho_n}.$$

Another one is

$$\chi(G) \leq 1 + \rho_1$$

with equality if and only if G is a complete graph or an odd cycle.

N.L. Biggs, Algebraic graph theory, Cambridge Univ. Press, Cambridge, 1974.

H.S. Wilf, The eigenvalues of a graph and its chromatic number, J. London Math. Soc. 42 (1967) 330–332.

P. Wocjan, C. Elphick, New spectral bounds on the chromatic number encompassing all eigenvalues of the adjacency matrix, Electron. J. Combin. 20 (2013) Paper 39.

Least (adjacency) eigenvalue

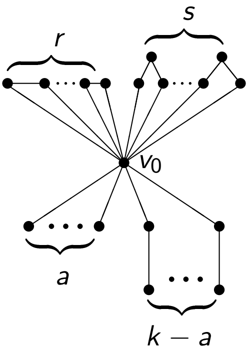
A connected graph is a cactus if any two of its cycles share at most one common vertex.

Petrović, Aleksić and Simić (2011) determined the unique cactus whose least eigenvalue is minimal among the cacti with n vertices and k cycles, where $0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$.

$T_{n,p}$: the tree consisting of p pendant paths at a common vertex with almost equal lengths.

Least (adjacency) eigenvalue

$C_{n,k}(r,s;a)$ with $2r+3s-a=n-2k-1$, $1\le k\le n-3$, and if $r\ge 1$, then $a=k$.



The graph $C_{n,k}(r,s;a)$.

Least (adjacency) eigenvalue

Theorem (Xing, Z)

Let G be a graph with minimum least eigenvalue among the n-vertex cacti with k pendant vertices, where $1 \le k \le n-1$. Let $n_0(k) = 18$ for $1 \le k \le 3$, $n_0(k) = 17$ for $4 \le k \le 7$, and $n_0(k) = 16$ for $8 \le k \le 10$. Then for $k = n-1, n-2$, $G \cong T_{n,k}$, and for $1 \le k \le n-3$, we have

(i) if $1 \le k \le 10$ and $n < n_0(k)$, then

(a) for $n-k \equiv 0 \pmod 3$, $G \cong C_{n,k}\left(1, \frac{n-k-3}{3}; k\right)$,

(b) for $n-k \equiv 1 \pmod 3$, $G \cong C_{n,k}\left(0, \frac{n-k-1}{3}; k\right)$,

(c) for $n-k \equiv 2 \pmod 3$, $G \cong C_{n,k}(0, \frac{n-k-2}{3}; k-1)$ if $(n,k) = (6,1), (9,1), (7,2), (8,3)$, or $(9,4)$, and $G \cong C_{n,k}\left(2, \frac{n-k-5}{3}; k\right)$ otherwise;

(ii) if $1 \le k \le 10$ and $n \ge n_0(k)$, or $k \ge 11$, then

(a) for $n-k \equiv 0 \pmod 2$, $G \cong C_{n,k}\left(\frac{n-k-4}{2}, 1; k\right)$,

(b) for $n-k \equiv 1 \pmod 2$, $G \cong C_{n,k}\left(\frac{n-k-1}{2}, 0; k\right)$.

Least (adjacency) eigenvalue

Theorem (Liu, Z)

Let G be a graph with minimum least eigenvalue among the n-vertex bicyclic graphs with k pendant vertices, where $1 \leq k \leq n-1$. Then ....

(Adjacency) eigenvalues

For a graph G with $U \subset V(G)$ and $F \subseteq E(G)$, Li et al. gave a new type lower bound for $\rho_1(G - U)$ in terms of $\rho_1(G)$ and the entries of the Perron vector of G. Mieghem et al. gave lower and upper bounds for $\rho_1(G - F)$ in terms of $\rho_1(G)$ and the entries of the Perron vector of G and $G - F$.

C. Li, H. Wang, P. Van Mieghem, Bounds for the spectral radius of a graph when nodes are removed, Linear Algebra Appl. 437 (2012) 319 – 323.

P. Van Mieghem, D. Stevanović, F.A. Kuipers, C. Li, R. van de Bovenkamp, D. Liu, H. Wang, Decreasing the spectral radius of a graph by link removals, Phys. Rev. E 84 (1) (2011) 016101

Least (adjacency) eigenvalue

Theorem (Xing, Z)

Let G be a graph with x being a least eigenvector. For $U \subset V(G)$, we have

$$\rho_{\min}(G) \leq \rho_{\min}(G - U) \leq \left(1 - 2\sum_{i \in U} x_i^2\right) \rho_{\min}(G) + 2 \sum_{\substack{\{i,j\} \subseteq U \\ i \sim j}} x_i x_j.$$

In particular, $\rho_{\min}(G) \leq \rho_{\min}(G - i) \leq (1 - 2x_i^2)\rho_{\min}(G)$, where $i \in V(G)$.

Least (adjacency) eigenvalue

Theorem (Xing, Z)

Let G be a graph with $F \subseteq E(G)$. Let x and y be least eigenvectors of G and $G - F$, respectively. Then

$$2 \sum_{ij \in F} x_i x_j \leq \rho_{\min}(G) - \rho_{\min}(G - F) \leq 2 \sum_{ij \in F} y_i y_j.$$

(Adjacency) eigenvalues — Estrada index

The Estrada index of a graph G is defined as

$$EE(G) = \sum_{i=1}^{n} e^{\rho_i}.$$

J.A. de la Peña, I. Gutman, J. Rada, Estimating the Estrada index, Linear Algebra Appl. 427 (2007) 70–76.

Let $m_k(G) = \sum_{i=1}^{n} \rho_i^k$ (number of closed walks of length k in G). Then $EE(G) = \sum_{k=0}^{\infty} \frac{M_k(G)}{k!}$.

$m_0(G) = n$, $m_1(G) = 0$, $m_2(G) = 2m$, $m_3(G) = 2t$.

Let G and H be two graphs of order n. For integer $k \geq 2$, let

$W_k(G)$ be the number of closed walk of length k in G.

If we can establish a injection σ_k from $W_k(G)$ to $W_k(H)$ for all k, then $M_k(G) \leq M_k(H)$, implying that $EE(G) \leq EE(H)$.

Moreover, if for some k_0, σ_{k_0} is not a surjection, $EE(G) < EE(H)$. $M_{k_0}(G) < M_{k_0}(H)$.

(Adjacency) eigenvalues — Estrada index

Theorem (Du,Z)

Let G be a tree with n vertices and p pendant vertices, where $2 \le p \le n-1$. Then $EE(G) \le EE(T_{n,p})$ with equality if and only if $G \cong T_{n,p}$.

For $2 \le r \le \lfloor n/2 \rfloor$, let $T^{n,r}$ be the tree obtained by attaching $r-1$ paths on two vertices to the center of the star S_{n-2r+2}.

Corollary

Let G be a tree with n vertices and matching number m, where $2 \le m \le \lfloor n/2 \rfloor$. Then $EE(G) \le EE(T^{n,m})$ with equality if and only if $G \cong T^{n,m}$.

Let G be a tree with n vertices and independence number α, where $\lceil n/2 \rceil \le \alpha \le n-2$. Then $EE(G) \le EE(T^{n,n-\alpha})$ with equality if and only if $G \cong T^{n,n-\alpha}$.

Let G be a tree with n vertices and domination number γ, where $2 \le \gamma \le \lfloor n/2 \rfloor$. Then $EE(G) \le EE(T^{n,\gamma})$ with equality if and only if $G \cong T^{n,\gamma}$

(Adjacency) eigenvalues — Estrada index

Let $D_{n,\Delta}$ be the tree obtained by adding an edge between the centers of two vertex-disjoint stars S_Δ, and attaching a path on $n - 2\Delta$ vertices to a pendant vertex, where $n \geq 2\Delta + 1 \geq 7$ [MATCH Commun. Math. Comput. Chem. 64 (2010) 799–810].

Theorem (Du,Z)

Let G be an n-vertex tree with two adjacent vertices of maximum degree Δ, where $n \geq 2\Delta + 1 \geq 7$. Then $EE(G) \geq EE(D_{n,\Delta})$ with equality if and only if $G \cong D_{n,\Delta}$.

(Adjacency) eigenvalues — Estrada index

Theorem (Du, Z)

Let G be a connected graph with n vertices and k cut edges, where $0 \leq k \leq n-3$. Then $EE(G) \leq EE(G_{n,k})$ with equality if and only if $G \cong G_{n,K}$, where $G_{n,k}$ is the graph obtained from the complete graph on $n-k$ vertices by attaching k pendant edges to a vertex.

(Adjacency) eigenvalues — Estrada index

Theorem (Du, Z)

(i) Let G be a unicyclic graph on $n \geq 4$ vertices. Then $EE(G) \leq EE(C_3(n-3))$ with equality if and only if $G \cong C_3(n-3)$.
(ii) Let G be an n-vertex unicyclic graph, where $n \geq 5$. If $G \ncong C_n, H_n$, then $EE(G) > \min\{EE(C_n), EE(H_n)\}$.

Laplacian eigenvalues — Sum of the first k Laplacian eigenvalues

Let $S_k(G) = \mu_1 + \cdots + \mu_k$ for a graph G with n vertices.

Let $d_i^*(G) = |\{v \in V(G) : d_v \geq i\}|$ for $i = 1, 2, \ldots, n$.

Grone–Merris conjecture (proven by Bai):

$$S_k(G) \leq \sum_{i=1}^{k} d_i^*(G) \text{ for } 1 \leq k \leq n.$$

R. Grone, R. Merris, The Laplacian spectrum of a graph II, SIAM J. Discrete Math. 7 (1994) 221–229.

H. Bai, The Grone-Merris conjecture, Trans. Amer. Math. Soc. 363 (2011) 4463–4474.

Laplacian eigenvalues — Sum of the first k Laplacian eigenvalues

As a variation of the Grone–Merris conjecture, Brouwer proposed the following Brouwer conjecture:

$$S_k(G) \leq e(G) + \binom{k+1}{2} \text{ for } 1 \leq k \leq n,$$

where $e(G)$ is the number of edges of G.

A.E. Brouwer, Spectra of graphs, Springer, New York, 2012.

Laplacian eigenvalues — Sum of the first k Laplacian eigenvalues

Brouwer: it is true for graphs with at most 10 vertices.

For $k = n - 1$ or n, it follows trivially because $S_k(G) = 2e(G)$.

For $k = 1$, it follows from the well-known inequality $\mu_1(G) \leq n$.

Haemers *et al.*: it is true for all graphs when $k = 2$.

Haemers *et al.*: it is true for trees.

W.H. Haemers, A. Mohammadian, B. Tayfeh-Rezaie, On the sum of Laplacian eigenvalues of graphs, Linear Algebra Appl. 432 (2010) 2214–2221.

Theorem

Brouwer conjecture is true for unicyclic graphs and bicyclic graphs.

Laplacian eigenvalues — Laplacian energy

The Laplacian energy of a graph G is defined as

$$LE(G) = \sum_{i=1}^{n} |\mu_i - \overline{d}|,$$

where $\overline{d}$ is the average degree of G.

Let α be the number of Laplacian eigenvalues at least the average $\overline{d}$. Then

$$LE(G) = 2S_\alpha(G) - 2\overline{d}\alpha.$$

Problems on Laplacian energy may be found in:

R. Brualdi, L. Hogben, B. Shader, AIM Workshop Spectra of Families of Matrices Described by Graphs, Digraphs, and Sign Patterns, Final Report: Mathematical Results (Revised). http://aimath.org/pastworkshops/matrixspectrumrep.pdf

For a graph G with order n and domination number $\gamma \geq 3$, Brand and Seifter showed that $\mu_1(G) < n - \lfloor \frac{\gamma-2}{2} \rfloor$.

C. Brand, N. Seifter, Eigenvalues and domination in graphs, Math. Slovaca 46 (1996) 33 – 39.

Let G be a bipartite graph with bipartition (U, W). Let G^+ be the set of graphs H such that $V(H) = V(G)$ and
$E(G) \subseteq E(H) \subseteq E(G) \cup E_U \cup E_W$, where
$E_U = \{uv : u, v \in U$ and $N_G(u : W) = N_G(v : W)\}$ and
$E_W = \{uv : u, v \in W$ and $N_G(u : U) = N_G(v : U)\}$. For $n \geq 4$, let $\mathcal{B}_n = \{K_{a,n-a} : 2 \leq a \leq \lfloor \frac{n}{2} \rfloor\}$.

Theorem

Let G be a graph with n vertices and domination number γ, where $2 \leq \gamma \leq n-1$. Then $\lambda_1(G) \leq n - \gamma + 2$ with equality if and only if $G \cong H \cup (\gamma-2)K_1$, where $H \in B^+$, $B \in \mathcal{B}_{n-\gamma+2}$ and $d_G(u) \leq n - \gamma$ for $u \in V(G)$.

Laplacian eigenvalues — Laplacian eigenvalues in some interval

Let T be a tree on $n \geq 2$ vertices. Braga, Rodrigues & Trevisan proved that $m_T[0,2) \geq \lceil \frac{n}{2} \rceil$. It is shown that $m_T[0,2) = \lceil \frac{n}{2} \rceil$ if and only if the matching number of T is $\lfloor \frac{n}{2} \rfloor$ (Zhou, Z, Du).

For a tree T, $m_T[0,2) \leq n - \gamma(T)$.

This was extended to all graphs without isolated vertices in:

D.M. Cardoso, D.P. Jacobs, V. Trevisan, Laplacian Distribution and Domination, Graphs Combin. 33 (2017) 1283–1295.

See also

S.T. Hedetniemi, D.P. Jacobs, V. Trevisan, Domination number and Laplacian eigenvalue distribution, Europ. J. Combin. 53 (2016) 66–71.

Laplacian eigenvalues — Laplacian coefficients

For a graph G on n vertices, let
$\det(xI_n - L(G)) = \sum_{k=0}^{n}(-1)^k c_k(G)x^{n-k}$.

Let T be a tree on n vertices. Gutman & Pavlović (2003) conjectured that

$$c_k(S_n) \leq c_k(T) \leq c_k(P_n) \text{ for } k = 1 \ldots, n,$$

and they showed that it is true for $k = 1, 2, 3, n-3, n-2, n-1, n$.

I. Gutman, L. Pavlović, On the coefficients of the Laplacian characteristic polynomial of trees, Bull. Acad. Serbe Sci. Arts 127 (2003) 31–40.

Z & Gutman (2008) showed that: Let T be a tree on n vertices and k an integer with $2 \leq k \leq n-2$. Then

$$c_k(T) > c_k(S_n) \text{ if } T \ncong S_n,$$

$$c_k(T) < c_k(P_n) \text{ if } T \ncong P_n.$$

'Zhou and Gutman recently proved that among all trees of order n, the kth coefficient c_k is largest when the tree is a path, and is smallest for stars. A new proof and a strengthening of this result is provided. A relation to the Wiener index is discussed.'

B. Mohar, On the Laplacian coefficients of acyclic graphs, Linear Algebra Appl. 722 (2007) 736–741.

Laplacian eigenvalues -Laplacian coefficients

'The first statement concerning the coefficients of the Laplacian polynomial was conjectured in [13] and was proved by Zhou and Gutman [25] by the aid of a surprising connection between the Laplacian polynomial and the adjacency polynomial of trees. A different proof was given by Mohar [19] using graph transformations. The same approach was used by Stevanović and Ilić [23] when they studied the extremal values of Laplacian coefficients of unicyclic graph.'

P. Csikvári, On a Poset of Trees II, J. Graph Theory 74 (2012) 81–103.

Signless Laplacian eigenvalues — Signless Laplacian spectral radius

Theorem (Fiedler & Nikiforov, 2010)

Let G be a graph on n vertices with $\overline{G}$.
(i) If $\rho_1(\overline{G}) \leq \sqrt{n-1}$ and $G \neq K_{n-1} + v$, then G contains a Hamiltonian path.
(ii) If $\rho_1(\overline{G}) \leq \sqrt{n-2}$ and $G \neq K_{n-1} + e$, then G contains a Hamiltonian cycle.

Let $\mathbb{EP}_n$ be the set of graphs of the following three types of graphs on n vertices: (a) a regular graph of degree $\frac{n}{2} - 1$, (b) a graph consisting of two complete components, and (c) the join of a regular graph of degree $\frac{n}{2} - 1 - r$ and a graph on r vertices, where $1 \leq r \leq \frac{n}{2} - 1$.

Let $\mathbb{EC}_n$ be the set of graphs of the following two types of graphs on n vertices: (a) the join of a trivial graph and a graph consisting of two complete components, and (b) the join of a regular graph of degree $\frac{n-1}{2} - r$ and a graph on r vertices, where $1 \leq r \leq \frac{n-1}{2}$.

Signless Laplacian eigenvalues — Signless Laplacian spectral radius

Theorem (2010)

Let G be a graph on n vertices with complement $\overline{G}$.
(i) If $\mu_1(\overline{G}) \leq n$ and $G \notin \mathbb{EP}_n$, then G contains a Hamiltonian path.
(ii) If $n \geq 3$, $\mu_1(\overline{G}) \leq n - 1$ and $G \notin \mathbb{EC}_n$, then G contains a Hamiltonian cycle.

Sketch of proof for (ii)

For an integer $k \geq 0$, the k-closure of the graph G is a graph obtained from G by successively joining pairs of nonadjacent vertices whose degree sum is at least k (in the resulting graph at each stage) until no such pair remains, denoted by $C_k(G)$.

Lemma 1 (Ore, 1960)

A graph G on n vertices has a Hamiltonian cycle if and only if $C_n(G)$ has one.

Lemma 2 (Ore, 1960)

Let G be a graph on n vertices. If $d_G(u) + d_G(v) \geq n$ for any pair of nonadjacent vertices u and v in G, then G contains a Hamiltonian cycle.

Signless Laplacian eigenvalue

For a graph G, let $Z(G) = \sum_{u \in V(G)} d_G(u)^2$. Obviously, $Z(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v))$.

Lemma 3

Let G be a graph with at least one edge. Then

$$\mu_1(G) \geq \frac{Z(G)}{m}$$

with equality if and only if the line graph $\mathcal{L}(G)$ of G is regular.

A semi–regular graph is a bipartite graph for which every vertex in the same partite set has the same degree.

For a connected graph G, $\mathcal{L}(G)$ is regular if and only if $d_G(u) + d_G(v)$ is a constant for any edge $uv \in E(G)$ if and only if G is regular or semi–regular.

Lemma 4 (Nash–Williams, 1969)

Every k-regular graph on $2k + 1$ vertices contains a Hamiltonian cycle, where $k \geq 2$.

Signless Laplacian eigenvalues

Let $H = C_n(G)$. Suppose that $H \ncong K_n$ and G has no Hamiltonian cycle . By Lemma 1, H has no Hamiltonian cycle. By Lemma 2 and the property of n-closure of G, $d_H(u) + d_H(v) \leq n-1$ for any pair of nonadjacent vertices u and v (always existing) in H. Thus

$$d_{\overline{H}}(u) + d_{\overline{H}}(v) \geq n-1 \text{ for any edge } uv \in E(\overline{H}).$$

It follows that

$$Z(\overline{H}) = \sum_{uv \in E(\overline{H})} \left(d_{\overline{H}}(u) + d_{\overline{H}}(v)\right) \geq (n-1)e(\overline{H}).$$

By Lemma 3, we have

$$\mu_1(\overline{H}) \geq \frac{Z(\overline{H})}{e(\overline{H})} \geq n-1.$$

Since $\overline{H}$ is a subgraph of $\overline{G}$, by Perron–Frobenius theorem,

$$\mu_1(\overline{G}) \geq \mu_1(\overline{H}) \geq n-1\,.$$

Since $\mu_1(\overline{G}) \leq n-1$, we have $\mu_1(\overline{G}) = \mu_1(\overline{H}) = \frac{Z(\overline{H})}{e(\overline{H})} = n-1$, and then $d_{\overline{H}}(u) + d_{\overline{H}}(v) = n-1$ for any $uv \in E(\overline{H})$, implying that $\overline{H}$ contains exactly one nontrivial component F, which is either regular or semi–regular, where $\frac{n+1}{2} \leq |V(F)| \leq n$.
Suppose that F is semi–regular. Then F contains at least $n-1$ vertices.
Claim. $\overline{H}$ is not connected.
By Claim, $\overline{H}$ consists of a complete bipartite graph F and an isolated vertex. Since $\mu_1(\overline{G}) = \mu_1(\overline{H})$, $\overline{H}$ is a subgraph of $\overline{G}$, by Perron–Frobenius theorem, $\overline{G} = \overline{H}$, and then G is the join of a trivial graph and a graph with two complete components, which contradicts the condition that G is not such a graph.

Thus F is a regular graph of degree $\frac{n-1}{2}$ that is not semi–regular.

If $F = \overline{H}$, thenby Perron–Frobenius theorem, $\overline{G} = \overline{H}$, and thus G $(= H)$ is a regular graph of degree $\frac{n-1}{2} - 1$, which contradicts Lemma 4. Thus $\overline{H}$ consists of F and additional $r = n - |V(F)|$ isolated vertices, where $1 \leq r \leq \frac{n-1}{2}$.

Note that $\mu_1(\overline{G}) = \mu_1(\overline{H})$ and $\overline{H}$ is a subgraph of $\overline{G}$. By Perron–Frobenius theorem, $\overline{G}$ consists of vertex–disjoint graph F and a graph F_1 on r vertices.

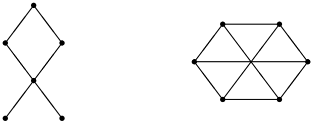
Thus G is the join of $\overline{F}$ (a regular graph of degree $\frac{n-1}{2} - 1 - r$) and $\overline{F_1}$ (a graph on r vertices), which contradicts the condition that G is not such a graph.

Signless Laplacian eigenvalues — Signless Laplacian eigenvalues in a interval

Let I be a real interval. Let $m_G I$ be the number of signless Laplacian eigenvalues belonging to I, multiplicities included.

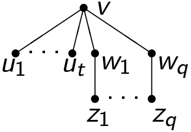
Theorem(Petrović, Gutman, Lepović & Milekić, 1999)

Let G be a connected bipartite graph. Then $m_G(3, +\infty) = 1$ if and only if
(i) $|V(G)| = 4, 5$, or
(ii) G is a spanning subgraph of F_1 or F_2, or
(iii) G is the graph $G(t, q)$ with $t, q \geq 0$ and $t + 2q + 1 = n \geq 6$.



F_1 F_2

Graphs F_1 and F_2.



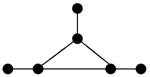
Graph $G(t, q)$.

Signless Laplacian eigenvalues

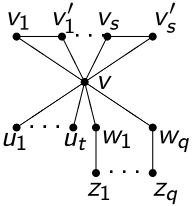
Theorem (Lin & Z, 2013)

Let G be a connected graph. Then $m_G(3, +\infty) = 1$ if and only if
(i) G is the triangle C_3, or
(ii) $|V(G)| = 4, 5$, or
(iii) G is a spanning subgraph of F_1 or F_2, or
(iv) G is the graph F, or
(v) G is the graph $G(s, t, q)$ for some s, t and q with $s, t, q \geq 0$, and $2s + t + 2q = n - 1$, where $n \geq 6$.

Signless Laplacian eigenvalues



Graph F.



Graph $G(s, t, q)$.

Extremal graphs on distance spectral radius

Among connected graphs on n vertices,
the complete graph achieves uniquely minimum distance spectral radius, the path achieves uniquely maximum distance spectral radius.

Among trees on n vertices,
the star achieves uniquely minimum distance spectral radius.

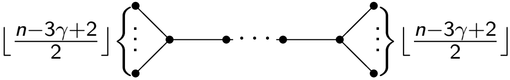
S.N. Ruzieh, D.L. Powers, The distance spectrum of the path P_n and the first distance eigenvector of connected graphs, Linear Multilinear Algebra 28 (1990) 75–81.

D. Stevanović, A. Ilić, Distance spectral radius of trees with fixed maximum degree, Electron. J. Linear Algebra 20 (2010) 168–179.

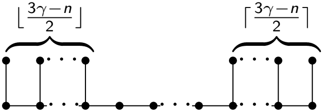
Distance spectral radius

Proposition (Stevanović & Ilić, 2010)

Let G be a non-trivial connected graph with $u \in V(G)$. For positive integers k and ℓ with $k \geq l$, let $G_u(k, \ell)$ be the graph obtained from G by attaching two pendant paths of length k and ℓ respectively at u, and $G_u(k, 0)$ the graph obtained from G by attaching a pendant path of length k at u. If $k \geq \ell \geq 1$, then $\rho(G_u(k, l)) < \rho(G_u(k+1, \ell-1))$.



$D\left(n, \lceil \frac{n-3\gamma+2}{2} \rceil, \lfloor \frac{n-3\gamma+2}{2} \rfloor\right)$

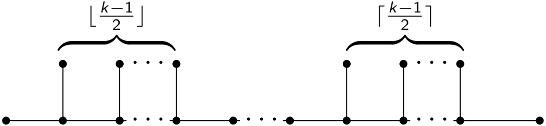


$E\left(n, \lfloor \frac{3\gamma-n}{2} \rfloor, \lceil \frac{3\gamma-n}{2} \rceil\right)$

Theorem

Among connected graphs with n vertices and domination number γ, where $1 \leq \gamma \leq \lfloor \frac{n}{2} \rfloor$, $D\left(n, \left\lceil \frac{n-3\gamma+2}{2} \right\rceil, \left\lfloor \frac{n-3\gamma+2}{2} \right\rfloor\right)$ for $1 \leq \gamma < \lceil \frac{n}{3} \rceil$, $E\left(n, \left\lfloor \frac{3\gamma-n}{2} \right\rfloor, \left\lceil \frac{3\gamma-n}{2} \right\rceil\right)$ for $\lceil \frac{n}{3} \rceil < \gamma \leq \lfloor \frac{n}{2} \rfloor$ are the unique graphs with maximum distance spectral radius.

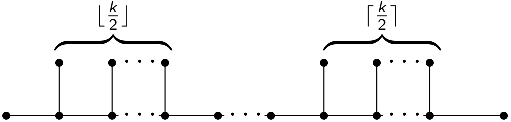
Theorem



$C_1\left(n, \lfloor \frac{k-1}{2} \rfloor, \lceil \frac{k-1}{2} \rceil\right)$

Theorem

Among trees with n vertices and $2k$ odd vertices, where $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, $C_1\left(n, \lfloor \frac{k-1}{2} \rfloor, \lceil \frac{k-1}{2} \rceil\right)$ is the unique tree with maximum distance spectral radius.

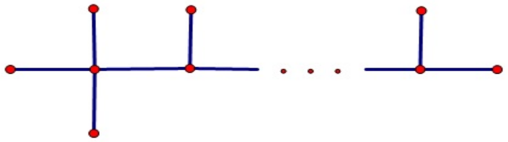


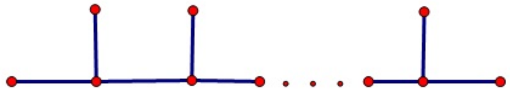
$C_1\left(n, \lfloor \frac{k}{2} \rfloor, \lceil \frac{k}{2} \rceil\right)$

Theorem

Let T be a tree with n vertices and k vertices of degree at least 3, where $0 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$. Then $\rho(T) \leq \rho\left(C_1\left(n, \lfloor \frac{k}{2} \rfloor, \lceil \frac{k}{2} \rceil\right)\right)$ with equality if and only if $T \cong C_1\left(n, \lfloor \frac{k}{2} \rfloor, \lceil \frac{k}{2} \rceil\right)$.

For an odd integer n, let F_n ne the tree displayed as following:





Theorem

Let T be a tree with maximum distance spectral radius among homeomorphically irreducible trees of order $n \geq 4$. Then $T \cong F_n$.

A sketch of proof for even n

The result is trivial if $n = 4$ and it may be checked for $n = 6$. Suppose $n \geq 8$.

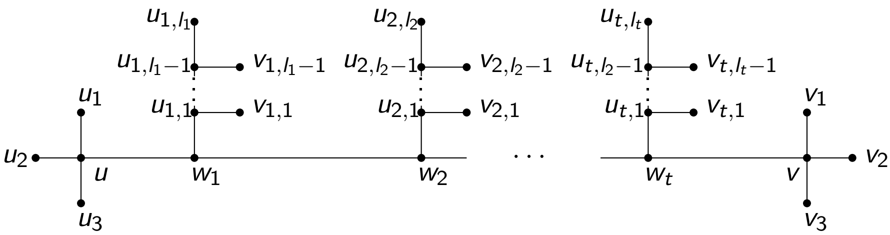
Claim. $\Delta(T) \leq 4$, and the number of vertices of degree 4 in T is 0 or 2.

Case 1. The number of vertices of degree 4 in T is 0. Then $k_1 = \frac{n-2}{2}$ and $k_3 = \frac{n+2}{2}$. First we show that T is a caterpillar. Next we show that $T \cong F_n$.

Case 2. The number of vertices of degree 4 in T is 2. Let $u, v \in V(T)$ be vertices of degree 4, $N_T(u) = \{u_1, u_2, u_3, u_4\}$ and $N_T(v) = \{v_1, v_2, v_3, v_4\}$. Let T_i be the component of $T - u$ containing u_i, where $1 \leq i \leq 4$. Suppose that there are two nontrivial T_i for $i = 1, 2, 3, 4$, say T_1 and T_2. Then $\delta_T(u_i) = 3$ for $i = 1, 2$. Suppose without loss of generality that $\sigma(T_1) \geq \sigma(T_2)$.

Let $T' = T - u_3u + u_3u_2$. Obviously, T' is an HIT. We may show that $\rho(T') > \rho(T)$, a contradiction. Thus there are exactly three trivial components of $T - u$. Similarly, there are exactly three trivial components of $T - v$. Note that there is a unique path connecting u and v.

Let $F(n; l_1, l_2, \ldots, l_t)$ be the graph as follows (with all vertices labelled). There are positive integers t, $l_1, \ldots, l_t$ such that $T \cong F(n; l_1, l_2, \ldots, l_t)$, where $n = 8 + \sum_{i=1}^{t} 2l_i$, and $uw_1 \ldots w_t v$ is the path connecting u and v in T, where $w_1 = u_4$ and $w_t = v_4$.



$\cdots$

Finally, we have $l_i = 1$ for all $1 \leq i \leq t$, which implies that $T \cong F(n; \underbrace{1, \ldots, 1}_{\frac{n-8}{2}})$.

Combining Cases 1 and 2, we have $T \cong F_n$ or $F(n; \underbrace{1, \ldots, 1}_{\frac{n-8}{2}})$. By showing that $\rho(F_n) > \rho(F(n; \underbrace{1, \ldots, 1}_{\frac{n-8}{2}}))$, we conclude that $T \cong F_n$.

Distance spectral radius of hypergraphs

Watanabe et al. studied some spectral properties of the distance matrix of a uniform hypertree.

S. Watanabe, K. Ishi, M. Sawa, A Q-analogue of the addressing problem of graphs by Graham and Pollak, SIAM J. Discrete Math. 26 (2012) 527–536.

Second largest distance eigenvalue

References

• B. Liu, B. Zhou, On the third largest eigenvalue of a graph, Linear Algebra Appl. 317 (2000) 193-200.

• B. Zhou, The changes in indices of modified graphs, Linear Algebra Appl. 356 (2002) 95-101.

• B. Zhou, N. Trinajstić, A note on Kirchhoff index, Chem. Phys. Lett. 445 (2008) 120-123.

• B. Zhou, I. Gutman, A connection between ordinary and Laplacian spectra of bipartite graphs, Linear and Multilinear Algebra 56 (2008) 305-310.

• B. Zhou, N. Trinajstić, On resistance-distance and Kirchhoff index, J. Math. Chem. 46 (2009) 283-289.

• B. Zhou, Signless Laplacian spectral radius and Hamiltonicity, Linear Algebra Appl. 432 (2010) 566-570.

References

• S. Wang, B. Zhou, The signless Laplacian spectra of the corona and edge corona of two graphs, Linear and Multilinear Algebra 61 (2013) 197-204.

• X. Duan, B. Zhou, Sharp bounds on the spectral radius of a nonnegative matrix, Linear Algebra Appl. 439 (2013) 2961-2970.

• J. Lan, B. Zhou, Spectra of graph operations based on R-graph, Linear and Multilinear Algebra 63 (2015) 1401–1422.

• L. Zhou, B. Zhou, Z. Du, On the number of Laplacian eigenvalues of trees smaller than two, Taiwanese J. Math. 19 (2015) 65–75.

• Y. Wang, B. Zhou, On distance spectral radius of graphs, Linear Algebra Appl. 438 (2013) 3490-3503.

• Y. Wang, R. Xing, B. Zhou, F. Dong, A note on distance spectral radius of trees, Spec. Matrices 5 (2017) 296–300.

• Y. Wang, B. Zhou, On distance spectral radius of hypergraphs, Linear Multilinear Algebra, https://doi.org/10.1080/03081087.2017.1391167.

References

- Z. Du, B. Zhou, The Estrada index of trees, Linear Algebra Appl. 435 (2011) 2462-2467.
- Z. Du, B. Zhou, On the Estrada index of graphs with given number of cut edges, Electron. J. Linear Algebra 22 (2011) 586-592.
- Z. Du, B. Zhou, The Estrada index of unicyclic graphs, Linear Algebra Appl. 436 (2012) 3149-3159.

- Z. Du, B. Zhou, The Estrada index of unicyclic graphs, Linear Algebra Appl. 436 (2012) 3149-3159.
- Z. Du, B. Zhou, The Estrada index of unicyclic graphs, Linear Algebra Appl. 436 (2012) 3149-3159.

- Z. Du, B. Zhou, Upper bounds for the sum of Laplacian eigenvalues of graphs, Linear Algebra Appl. 436 (2012) 3672-3683.
- Z. Du, B. Zhou, R. Xing, On maximum Estrada indices of graphs with given parameters, Linear Algebra Appl. 436 (2012) 3767-3772.

References

- Z. Liu, B. Zhou, On least eigenvalue of bicyclic graphs with fixed number of pendant vertices, J. Math. Sci. (N. Y.) 182 (2012) 175-192; Translated from Sovrem. Mat. Prilozh. (Contemporary Mathematics and Its Applications), Vol. 71, Algebraic Techniques in Graph Theory and Optimization, 2011.
- R. Xing, B. Zhou, On the least eigenvalue of cacti with pendant vertices, Linear Algebra Appl. 438 (2013) 2256-2273.
- R. Xing, B. Zhou, On least eigenvalues and least eigenvectors of real symmetric matrices and graphs, Linear Algebra Appl. 438 (2013) 2378-2384.
- R. Xing, B. Zhou, On the distance and distance signless Laplacian spectral radii of bicyclic graphs, Linear Algebra Appl. 439 (2013) 3955-3963.
- R. Xing, B. Zhou, Sharp bounds for the spectral radius of nonnegative matrices, Linear Algebra Appl. 449 (2014) 194-209.
- R. Xing, B. Zhou, J. Li, On the distance signless Laplacian spectral radius of graphs, Linear and Multilinear Algebra 62 (2014) 1377-1387.
- R. Xing, B. Zhou, F. Dong, The effect of a graft transformation on distance spectral radius, Linear Algebra Appl. 457 (2014) 261-275.
- R. Xing, B. Zhou, Laplacian and signless Laplacian spectral radii of graphs with fixed domination number, Math. Nachr. 288 (2015) 476-480.

References

• H. Lin, B. Zhou, On least distance eigenvalues of trees, unicyclic graphs and bicyclic graphs, Linear Algebra Appl. 443 (2014) 153-163.

• H. Lin, B. Zhou, Graphs with at most one signless Laplacian eigenvalue exceeding three, Linear and Multilinear Algebra 63 (2015) 377-383.

• H. Lin, B. Zhou, On sharp bounds for spectral radius of nonnegative matrices, Linear Multilinear Algebra 65 (2017) 1554–1565.

• H. Lin, B. Zhou, Spectral radius of uniform hypergraphs, Linear Algebra Appl. 527 (2017) 32–52.

• H. Lin, B. Zhou, Y. Wang, Distance (signless) Laplacian spectral radius of uniform hypergraphs, Linear Algebra Appl. 529 (2017) 271–293.

• H. Lin, B. Zhou, Yaduan Li, On distance spectral radius of uniform hypergraphs, Linear Multilinear Algebra 66 (2018) 497–513.

• H. Lin, B. Zhou, On distance spectral radius of uniform hypergraphs with cycles, Discrete Appl. Math. 239 (2018) 125–143.

• H. Lin, Zhibin Du, B. Zhou, On the first two largest distance Laplacian eigenvalues of unicyclic graphs, Linear Algebra Appl. 546 (2018) 289–307.

• H. Lin, B. Zhou, The distance spectral radius of trees, Linear Multilinear Algebra, https://doi.org/10.1080/03081087.2017.1418830.

Thank you!