Sharp upper and lower bounds for the spectral radius of a nonnegative weakly irreducible tensor and its application

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July 7, 2018

Outline of the Talk

1. Part I : Matrix and its spectrum
2. Part II : Tensor and its spectrum
3. Part III : Applications to a k-uniform hypergraph

Part I : Matrix and its spectrum

Definitions and notations

- M: a real matrix of order n.
- $\lambda_1, \lambda_2, \ldots, \lambda_n$: the eigenvalues of M with $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$.
- $\rho(M)$: the spectral radius of M, $\rho(M) = |\lambda_1|$.
- If M is a nonnegative matrix, it follows from the Perron-Frobenius theorem that the spectral radius $\rho(M)$ is a eigenvalue of M.
- If M is a nonnegative irreducible matrix, it follows from the Perron-Frobenius theorem that $\rho(M) = \lambda_1$ is simple.

Definitions and notations

Let G be a graph.

- $A(G) = (a_{ij})$: the adjacency matrix of G, where $a_{ij} = 1$ if v_i and v_j are adjacent and 0 otherwise.
- The spectral radius of $A(G)$: $\rho(G)$.
- $diag(G) = diag(d_1, d_2, \ldots, d_n)$: the diagonal matrix of vertex degrees of a graph G.
- The signless Laplacian matrix of G: $Q(G) = diag(G) + A(G)$.
- The spectral radius of $Q(G)$: $q(G)$.

Definitions and notations

- $\mathcal{D}(G) = (d_{ij})$: the distance adjacency matrix of G.
- The spectral radius of $\mathcal{D}(G)$: $\rho^{\mathcal{D}}(G)$.
- $Tr(G) = diag(D_1, D_2, \ldots, D_n)$: the diagonal matrix of vertex transmission of G.
- The distance signless Laplacian matrix of G: $\mathcal{Q}(G) = Tr(G) + \mathcal{D}(G)$.
- The spectral radius of $\mathcal{Q}(G)$: $q^{\mathcal{D}}(G)$.

Definitions and notations

Let $\overrightarrow{G}$ be a digraph.

- $A(\overrightarrow{G}) = (a_{ij})$: the adjacency matrix of $\overrightarrow{G}$, where a_{ij} is equal to the number of arc (v_i, v_j).
- The spectral radius of $A(\overrightarrow{G})$: $\rho(\overrightarrow{G})$.
- $diag(\overrightarrow{G}) = diag(d_1^+, d_2^+, \ldots, d_n^+)$: the diagonal matrix of vertex out-degrees of $\overrightarrow{G}$.
- The signless Laplacian matrix of $\overrightarrow{G}$: $Q(\overrightarrow{G}) = diag(\overrightarrow{G}) + A(\overrightarrow{G})$.
- The spectral radius of $Q(\overrightarrow{G})$: $q(\overrightarrow{G})$.

Notation of a graph

Let $G = (V, E)$ be a simple (connected) graph with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G)$.

- d_i: the degree of vertex v_i.
- $i \sim j$: v_i is adjacent to v_j.
- $m_i = \frac{\sum_{i \sim j} d_j}{d_i}$: the average degree of the neighbors of v_i in G.
- d_{uv}: the distance between u and v, is the length of the shortest path connecting them in G.
- $D_i = \sum_{j=1}^{n} d_{ij}$: the distance degree of vertex v_i in G, or the transmission of vertex v_i in G.
- $T_i = \sum_{j=1}^{n} d_{ij} D_j$: the second distance degree of vertex v_i in G.

Some known results

Theorem 1

Let $G=(V,E)$ be a simple connected graph on n vertices. Then

$$\rho(G) \leq \max_{1\leq i,j\leq n} \{\sqrt{m_i m_j}, i \sim j\}.$$

Moreover, the equality holds if and only if one of the following two conditions holds:
(1) $m_1 = m_2 = \ldots = m_n$;
(2) G is a bipartite graph and the vertices of same partition have the same average degree.

K.C. Das, P. Kumar, Some new bounds on the spectral radius of graphs, Discrete Math. 281 (2004) 149–161.

Some known results

Theorem 2

Let $G=(V,E)$ be a connected graph on n vertices, for any $1 \le i, j \le n$, $g(i,j)=\frac{d_i+d_j+\sqrt{(d_i-d_j)^2+4m_im_j}}{2}$. Then we have

$$\min\{g(i,j), i \sim j\} \le q(G) \le \max\{g(i,j), i \sim j\},$$

and one of the equalities holds if and only if one of the following conditions holds:

(1) G is a regular graph;

(2) G is a bipartite semi-regular graph;

A.D. Maden, K.C. Das, A.S. Cevik, Sharp upper bounds on the spectral radius of the signless Laplacian matrix of a graph, Appl Math Comput. 219 (2013) 5025–5032.

Some known results

Theorem 3

Let $G = (V, E)$ be a connected graph on n vertices. Then

$$\rho^{\mathcal{D}}(G) \le \max_{1\le i,j\le n}\left\{\sqrt{\frac{T_iT_j}{D_iD_j}}\right\},$$

and the equality holds if and only if $\frac{T_1}{D_1} = \frac{T_2}{D_2} = \ldots = \frac{T_n}{D_n}$.

C.X. He, Y. Liu, Z.H. Zhao, Some new sharp bounds on the distance spectral radius of graph, MATCH Commum. Math. Comput. Chem. 63 (2010) 783–788.

Some known results

Theorem 4 (Hong and Y., 2014, AMC)

Let $G = (V, E)$ be a connected graph on n vertices, for all $1 \leq i, j \leq n$, $h(i,j) = \frac{D_i + D_j + \sqrt{(D_i - D_j)^2 + \frac{4T_iT_j}{D_iD_j}}}{2}$. Then

$$q^{\mathcal{D}}(G) \leq \max_{1 \leq i,j \leq n} \{h(i,j)\},$$

and the equality holds if and only if $D_1 + \frac{T_1}{D_1} = \ldots = D_n + \frac{T_n}{D_n}$.

W.X. Hong, L.H. You, Futher results on the spectral radius of matrices and graphs, Applied Math Comput. 239 (2014) 326–332.

Notation of a digraph

Let $\overrightarrow{G} = (V, E)$ be a digraph on n vertices.

- $i \sim j$: $(v_i, v_j) \in E$.
- d_i^+: the out-degree of the vertex v_i in $\overrightarrow{G}$.
- $m_i^+ = \frac{\sum\limits_{i \sim j} d_j^+}{d_i^+}$: the average out-degree of the out-neighbors of v_i in $\overrightarrow{G}$.

Some known results

Theorem 5

Let $\overrightarrow{G} = (V, E)$ be a strong connected digraph on n vertices. Then

$$\min_{1\le i,j\le n} \{\sqrt{m_i^+ m_j^+}, i \sim j\} \le \rho(\overrightarrow{G}) \le \max_{1\le i,j\le n} \{\sqrt{m_i^+ m_j^+}, i \sim j\},$$

and one of the equalities holds if and only if one of the following two conditions holds:

(1) $m_1^+ = m_2^+ = \ldots = m_n^+$,

(2) $\overrightarrow{G}$ is a bipartite graph and the vertices of same partition have the same average out-degree.

G.H. Xu, C.Q. Xu, Sharp bounds for the spectral radius of digraphs, Linear Algebra Appl. 430 (2009) 1607–1612.

Some known results

Theorem 6

Let $\overrightarrow{G} = (V, E)$ be a strong connected digraph on n vertices, $G(i,j) = \frac{d_i^+ + d_j^+ + \sqrt{(d_i^+ - d_j^+)^2 + 4m_i^+ m_j^+}}{2}$ for any $i, j \in \{1, 2, \ldots, n\}$. Then

$$\min_{1 \le i,j \le n} \{G(i,j), i \sim j\} \le q(\overrightarrow{G}) \le \max_{1 \le i,j \le n} \{G(i,j), i \sim j\}.$$

S.B. Bozkurt, D. Bozkurt, On the signless Laplacian spectral radius of digraphs, Ars Combin. 108 (2013) 193–200.

Our results

Theorem 7 (Y., Shu and Yuan, 2017, LAMA)

Let $A = (a_{ij})$ be an $n \times n$ nonnegative irreducible matrix with $a_{ii} = 0$ for $i = 1, 2, \ldots, n$, and the row sum $r_1, r_2, \ldots, r_n$. Let $B = A + M$, where $M = diag(t_1, t_2, \ldots, t_n)$ with $t_i \geq 0$ for any $i \in \{1, 2, \ldots, n\}$, $s_i = \sum_{j=1}^{n} a_{ij} r_j$, $\rho(B)$ be the spectral radius of B.

Let $f(i,j) = \frac{t_i + t_j + \sqrt{(t_i - t_j)^2 + \frac{4 s_i s_j}{r_i r_j}}}{2}$ for any $1 \leq i, j \leq n$. Then

$$\min_{1 \leq i,j \leq n} \{f(i,j), a_{ij} \neq 0\} \leq \rho(B) \leq \max_{1 \leq i,j \leq n} \{f(i,j), a_{ij} \neq 0\}. \quad (1)$$

L.H. You, Y.J. Shu, P.Z. Yuan, Sharp upper and lower bounds for the spectral radius of a nonnegative irreducible matrix and its applications, Linear Multilinear Algebra, 65(1) (2017).

(Cont.) Moreover, the equalities in (1) hold if and only if one of the two conditions holds:

(i) $t_i + \frac{s_i}{r_i} = t_j + \frac{s_j}{r_j}$ for any $i, j \in \{1, 2, \ldots, n\}$;

(ii) There exist nonempty proper subsets U and W of $[n]$ such that

1) $[n] = U \cup W$ with $U \cap W = \phi$,

2) $a_{ij} \neq 0$ only when $i \in U, j \in W$ or $i \in W, j \in U$,

3) there exists $\ell > 0$ such that $\rho(\mathbb{B}) = t_i + \frac{\ell s_i}{r_i} = t_j + \frac{s_j}{\ell r_j}$ for all $i \in U$ and $j \in W$. In fact, $\ell > 1$ when the left equality holds and $\ell < 1$ when the right equality holds.

Part II : Tensor and its spectrum

Tensor (or hypermatrix, or multiple array)

Definition

An order m dimension n tensor (or hypermatrix, or multiple array) $\mathcal{T} = (t_{i_1 i_2 \cdots i_m})$ $(1 \leq i_j \leq n, j = 1, \cdots, m)$ over the complex field $\mathbb{C}$ is a multidimensional array with all entries $t_{i_1 i_2 \cdots i_m} \in \mathbb{C}$.

A general product of tensors

Let $\mathcal{A}$ (and $\mathcal{B}$) be order $m \geq 2$ (and $k \geq 1$) dimension n tensor, respectively. The product $\mathcal{AB}$ is the following tensor $\mathcal{C}$ of dimension n and order $(m-1)(k-1)+1$ with entries:

$$\mathcal{C}_{i\alpha_1\cdots\alpha_{m-1}} = \sum_{i_2,\cdots,i_m\in[n_2]} \mathcal{A}_{ii_2\cdots i_m}\mathcal{B}_{i_2\alpha_1}\cdots\mathcal{B}_{i_m\alpha_{m-1}},$$

where $i \in [n], \alpha_1, \cdots, \alpha_{m-1} \in [n]^{k-1}$.

J. Shao, A general product of tensors with applications, Linear Algebra Appl., 439(2013)2350-2366.

Eigenequations, eigenvalues and eigenvectors of a tensor

Let $\mathcal{T}$ be an order m dimension n tensor, and $x = (x_1, \cdots, x_n)^T \in \mathbb{C}^n$ be a column vector of dimension n. By the rules of general product of tensors defined by Shao, $\mathcal{T}x$ is a vector in $\mathbb{C}^n$ whose ith component is as the following

$$(\mathcal{T}x)_i = \sum_{i_2,\cdots,i_m=1}^{n} t_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m}.$$

Eigenequations, eigenvalue and eigenvector

Denote by $x^{[r]} = (x_1^r, \cdots, x_n^r)^T$.
A number $\lambda \in \mathbb{C}$ is called an eigenvalue of the m-th order tensor $\mathcal{T}$ if there exists a nonzero vector $x \in \mathbb{C}^n$ satisfying the following eigenequations

$$\mathcal{T}x = \lambda x^{[m-1]},$$

and in this case, x is called an eigenvector of $\mathcal{T}$ corresponding to eigenvalue λ.

K.C. Chang, K. Pearson, T. Zhang, Perron-Frobenius theorem for nonnegative tensors, Commun. Math. Sci. 6(2008)507-520.

The spectral radius for a tensor

Let $\mathcal{T}$ be an order m dimension n tensor.

- $\sharp$ eigenvalues of $\mathcal{T} = n(m-1)^{n-1}$.
- The spectral radius of $\mathcal{T}$:
  $\rho(\mathcal{T}) = max\{|\mu| : \mu$ is an eigenvalue of $\mathcal{T}\}$.

K.C. Chang, K. Pearson, T. Zhang, Perron-Frobenius theorem for nonnegative tensors, Comm. Math. Sci., 2008, 6 (2): 507-520.

L.Q. Qi, Eigenvalues of a real supersymmetric tensor, J. Symb. Comput., 2005, 40: 1302-1324.

Sharp upper and lower bounds

Theorem 8 (Y., Huang and Yuan, 2018+)

Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be a nonnegative weakly irreducible tensor with order m dimension n and $a_{i \cdots i} = 0$ for $i \in [n]$. Let $t_i \geq 0$, $R_i > 0$, and $S_i = \sum_{i_2, \ldots, i_m = 1}^{n} a_{i i_2 \ldots i_m} R_{i_2} \ldots R_{i_m}$ for any $i \in [n]$. Let $\mathcal{B} = \mathcal{A} + \mathcal{M}$, where $\mathcal{M}$ is a diagonal tensor with its diagonal element $m_{ii \cdots i}$ being t_i. For any $1 \leq i, j \leq n$, write

$$F(i,j) = \frac{t_i + t_j + \sqrt{(t_i - t_j)^2 + \frac{4 S_i S_j}{(R_i R_j)^{m-1}}}}{2}.$$

Then

$$\min_{1 \leq i,j \leq n} \{F(i,j), j \in N(i)\} \leq \rho(\mathcal{B}) \leq \max_{1 \leq i,j \leq n} \{F(i,j), j \in N(i)\}. \quad (2)$$

(Cont.)Moreover, one of the equalities in (2) holds if and only if one of the two conditions holds:

(i) $t_i + \frac{S_i}{R_i^{m-1}} = t_j + \frac{S_j}{R_j^{m-1}}$ for any $i, j \in [n]$.

(ii) There exist nonempty proper subsets U and W of $[n]$ such that

1) $[n] = U \cup W$ with $U \cap W = \phi$;

2) $a_{i_1 i_2 \ldots i_m} \neq 0$ only when $i_1 \in U, i_2, \ldots, i_m \in W$ or $i_1 \in W, i_2, \ldots, i_m \in U$;

3) there exists $\ell > 0$ such that $\rho(\mathbb{B}) = t_i + \frac{\ell^{m-1} S_i}{R_i^{m-1}} = t_j + \frac{S_j}{\ell^{m-1} R_j^{m-1}}$ for all $i \in U$ and $j \in W$. In fact, $\ell > 1$ when the left equality holds and $\ell < 1$ when the right equality holds.

Applications to tensors and matrices

- The i-th row sum of $\mathcal{A}$ is defined as $r_i(\mathcal{A}) = \sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\ldots i_m}$.
- Denote $N_{\mathcal{A}}(i)$ (or simply $N(i)$) by

$$N_{\mathcal{A}}(i) = \{i_2, \ldots, i_m | a_{ii_2\ldots i_m} \neq 0\}.$$

- Write $s_i = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} r_{i_2} \ldots r_{i_m}$.

Application 1: Let $R_i = r_i$ and $S_i = s_i$.

Application 2: Let $m = 2$, $R_i = r_i$ and $S_i = s_i$.

Then Theorem 7 is the corollary of Theorem 8.

Main strategy of the proof of Theorem 8

- Two diagonal similar tensors have the same spectrum. Consider $R^{-(m-1)}\mathcal{B}R$ instead of $\mathcal{B}$, where $R = diag(R_1, \ldots, R_n)$.

Lemma 1

Suppose that the two tensors $\mathcal{A}$ and $\mathcal{B}$ are diagonal similar, namely $\mathcal{B} = D^{-(m-1)}\mathcal{A}D$ for some invertible diagonal matrix D. Then x is an eigenvector of $\mathcal{B}$ corresponding to the eigenvalue λ if and only if $y = Dx$ is an eigenvalues of $\mathcal{A}$ corresponding to the same eigenvalue λ.

J. Shao, A general product of tensors with applications, Linear Algebra Appl., 439(2013)2350-2366.

Main strategy of the proof of Theorem 8

- Analyze the components of the Perron vector of a nonnegative weakly irreducible tensor.

Lemma 2

Let $\mathcal{A}$ be a nonnegative tensor of order m dimension n. If some eigenvalue of $\mathcal{A}$ has a positive eigenvector corresponding to it, then this eigenvalue must be $\rho(\mathcal{A})$.

Y.N. Yang, Q.Z. Yang, Further results for Perron-Frobenius theorem for nonnegative tensors, SIAM J. Matrix Anal.Appl. 31(5) (2010) 2517–2530.

Main strategy of the proof of Theorem 8

Lemma 3

Let $\mathcal{A}$ be a nonnegative tensor of order m dimension n. Then $\rho(\mathcal{A})$ is an H-eigenvalue of $\rho(\mathcal{A})$ with a nonnegagtive eigenvector. Furthermore, if $\mathcal{A}$ is weakly irreducible, then $\rho(\mathcal{A})$ has a positive eigenvector.

- S. Friedland, A. Gaubert, L. Han, Perron-Frobenius theorems for nonnegative multilinear forms and extensions, Linear Algebra Appl. 438 (2013), 738–749.
- Y.N. Yang, Q.Z. Yang, On some properties of nonnegative weakly irreducible tensors, arXiv: 1111.0713v3, 2011.

Main strategy of the proof of Theorem 8

• To prove the equalities cases, the (nonnegative matrix) representation $G(\mathcal{A})$ of a tensor $\mathcal{A}$, and the associated directed graph $D(G(\mathcal{A}))$ of a nonnegative matrix and its strongly connectedness will be used.

• $G(\mathcal{A})$ is the representation associated with the nonnegative tensor $\mathcal{A}$, if the (i, j)-th entry of $G(\mathcal{A})$ is defined to be the summation of $\mathcal{A}_{ii_2\cdots i_m}$ with indices $\{i_2, \cdots i_m\}$, where $j \in \{i_2, \cdots, i_m\}$.

• A nonnegative matrix A is irreducible if and only if its associated directed graph $D(A)$ is strongly connected.

Main strategy of the proof of Theorem 8

Proposition 1

Let $\mathcal{A}$ be a nonnegative tensor of order m and dimension n, $\mathrm{G}(\mathcal{A})$ be the representation associated matrix to $\mathcal{A}$, and $\mathrm{D}(\mathrm{G}(\mathcal{A}))$ be the associated directed graph of $\mathrm{G}(\mathcal{A})$. Then the following three conditions are equivalent:

(1) $\mathcal{A}$ is weakly irreducible.

(2) $\mathrm{G}(\mathcal{A})$ is irreducible.

(3) $\mathrm{D}(\mathrm{G}(\mathcal{A}))$ is strongly connected.

Part III : Applications to a k-uniform hypergraph

Hypergraph and k-uniform hypergraph

Definition

A hypergraph is a pair $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$, where $V(\mathcal{H})$ is the set of vertices of $\mathcal{H}$ and $E(\mathcal{H})$ is a family of non-empty subsets of $V(\mathcal{H})$.

Definition

A k- uniform hypergraph (or called k-graph) $\mathcal{H}$ consists of a set of vertices $V(\mathcal{H})$ and a set $E(\mathcal{H})$ of k-subsets of $V(\mathcal{H})$.

Tensors of uniform hypergraphs

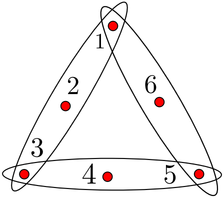
Definition

The adjacency tensor of a k-uniform hypergraph $\mathcal{H}$ with n vertices is defined as the k-th order n-dimensional tensor $\mathcal{A}(\mathcal{H})$ with

$$(\mathcal{A}(\mathcal{H}))_{i_1 i_2 \cdots i_k} = \begin{cases} \frac{1}{(k-1)!} & \{i_1, i_2, \cdots, i_k\} \in E(\mathcal{H}) \\ 0 & \text{otherwise.} \end{cases}$$

J. Cooper and A. Dutle, Spectra of uniform hypergraphs, Linear Algebra and Its Applications, 436 (2012) 3268-3292.

An example



H

$$
\begin{aligned}
(\mathcal{A}(\mathrm{H}))_{123} &= (\mathcal{A}(\mathrm{H}))_{132} = (\mathcal{A}(\mathrm{H}))_{213} = (\mathcal{A}(\mathrm{H}))_{231} \\
&= (\mathcal{A}(\mathrm{H}))_{312} = (\mathcal{A}(\mathrm{H}))_{321} = \frac{1}{2}, \\
&\cdots\cdots \\
(\mathcal{A}(\mathrm{H}))_{124} &= (\mathcal{A}(\mathrm{H}))_{142} = (\mathcal{A}(\mathrm{H}))_{214} = (\mathcal{A}(\mathrm{H}))_{241} \\
&= (\mathcal{A}(\mathrm{H}))_{412} = (\mathcal{A}(\mathrm{H}))_{421} = 0, \\
&\cdots\cdots
\end{aligned}
$$

Tensors of uniform hypergraphs

Definition

Let $\mathcal{D}(\mathcal{H})$ be a k-th order n-dimensional diagonal tensor with its diagonal entry $\mathcal{D}_{ii\cdots i}$ being d_i, the degree of vertex i, for all $i \in V(\mathcal{H}) = [n]$ and the other entries being 0. Then $\mathcal{Q}(\mathcal{H}) = \mathcal{D}(\mathcal{H}) + \mathcal{A}(\mathcal{H})$ is the signless Laplacian tensor of the hypergraph $\mathcal{H}$.

L. Qi, H^+-eigenvalue of Laplacian and signless Laplacian tensors, Commun. Math. Sci. 12 (2014), 1045–1064.

Tensors of uniform hypergraphs

Results

It was proved that a k-uniform hypergraph $\mathcal{H}$ is connected if and only if its adjacency tensor $\mathcal{A}(\mathcal{H})$ (and so $\mathcal{Q}(\mathcal{H})$ is weakly irreducible.

For a vertices i of k-uniform hypergraph $\mathcal{H}$, denoted $m_i = \frac{\sum_{\{i,i_2,\ldots,i_k\}\in E(\mathcal{H})} d_{i_2}\ldots d_{i_k}}{d_i^{k-1}}$, which is a generalization of the average of degrees of vertices adjacent to i of the ordinary graph.

S. Friedland, A. Gaubert, L. Han, Perron-Frobenius theorems for nonnegative multilinear forms and extensions, Linear Algebra Appl. 438 (2013), 738–749.

Applications of Theorem 8 to a connected k-uniform hypergraph

Take $\mathbb{A} = \mathbb{B} = \mathbb{A}(\mathcal{H})$ in Theorem 8. Then we have

Theorem 9

Let $k \geq 3$, $\mathcal{H}$ be a connected k-uniform hypergraph on n vertices and $R_i > 0$ for any $i \in [n]$. Then

$$\min_{e\in E(\mathcal{H})} \min_{\{i,j\}\subseteq e} \sqrt{R'_i R'_j} \leq \rho(\mathbb{A}(\mathcal{H})) \leq \max_{e\in E(\mathcal{H})} \max_{\{i,j\}\subseteq e} \sqrt{R'_i R'_j}, \qquad (3)$$

where $R'_i = R_i^{-(k-1)} \sum_{\{i,i_2,\ldots,i_k\}\in E(\mathcal{H})} R_{i_2} \ldots R_{i_k}$ for any $i \in [n]$.

Moreover, one of the equalities in (3) holds if and only if $R'_i = R'_j$ for any $i, j \in [n]$.

Applications of Theorem 8 to a connected k-uniform hypergraph

Take $R_i = d_i$. Then $R'_i = m_i$ for any $i \in [n]$.

Corollary 1

Let $\mathcal{H}$ be a connected k-uniform hypergraph on n vertices. Then

$$\min_{e\in E(\mathcal{H})} \min_{\{i,j\}\subseteq e} \sqrt{m_i m_j} \leq \rho(\mathbb{A}(\mathcal{H})) \leq \max_{e\in E(\mathcal{H})} \max_{\{i,j\}\subseteq e} \sqrt{m_i m_j}. \quad (4)$$

Moreover, one of the equalities in (4) holds if and only if $m_i = m_j$ for any $i, j \in [n]$.

X.Y. Yuan, M. Zhang, M. Lu, Some upper bounds on the eigenvalues of uniform hypergraphs, Linear Algebra Appl. 484 (2015), 540–549.

Applications of Theorem 8 to a connected k-uniform hypergraph

Take $R_i = 1$. Then $R'_i = d_i$ for any $i \in [n]$.

Corollary 2

Let $\mathcal{H}$ be a connected k-uniform hypergraph on n vertices. Then

$$\min_{e\in E(\mathcal{H})} \min_{\{i,j\}\subseteq e} \sqrt{d_i d_j} \leq \rho(\mathbb{A}(\mathcal{H})) \leq \max_{e\in E(\mathcal{H})} \max_{\{i,j\}\subseteq e} \sqrt{d_i d_j}. \qquad (5)$$

Moreover, one of the equalities in (5) holds if and only if $\mathcal{H}$ is a regular hypergraph.

X.Y. Yuan, M. Zhang, M. Lu, Some upper bounds on the eigenvalues of uniform hypergraphs, Linear Algebra Appl. 484 (2015), 540–549.

Applications of Theorem 8 to a connected k-uniform hypergraph

Take $\mathbb{A} = \mathbb{A}(\mathcal{H})$ and $\mathbb{B} = \mathbb{Q}(\mathcal{H})$. Then

Theorem 10

Let $k \geq 3$, $\mathcal{H}$ be a connected k-uniform hypergraph on n vertices and $b_i > 0$ for any $i \in [n]$. Then

$$\min_{e\in E(\mathcal{H})} \min_{\{i,j\}\subseteq e} g(i,j) \leq \rho(\mathbb{Q}(\mathcal{H})) \leq \max_{e\in E(\mathcal{H})} \max_{\{i,j\}\subseteq e} g(i,j), \tag{6}$$

where $g(i,j) = \frac{d_i+d_j+\sqrt{(d_i-d_j)^2+4R_i'R_j'}}{2}$ and $R_i' = R_i^{-(k-1)} \sum_{\{i,i_2,\ldots,i_k\}\in E(\mathcal{H})} R_{i_2}\ldots R_{i_k}$ for any $i \in [n]$.

Moreover, one of the equalities in (6) holds if and only if $d_i + R_i' = d_j + R_j'$ for any $i, j \in [n]$.

Applications of Theorem 8 to a connected k-uniform hypergraph

Let $R_i = d_i$. Then $R'_i = m_i$ for any $i \in [n]$.

Corollary 3

Let $\mathcal{H}$ be a connected k-uniform hypergraph on n vertices. Then

$$\min_{e\in E(\mathcal{H})} \min_{\{i,j\}\subseteq e} G(i,j) \le \rho(\mathbb{Q}(\mathcal{H})) \le \max_{e\in E(\mathcal{H})} \max_{\{i,j\}\subseteq e} G(i,j), \qquad (7)$$

where $G(i,j) = \frac{d_i+d_j+\sqrt{(d_i-d_j)^2+4m_i m_j}}{2}$. Moreover, one of the equalities in (7) holds if and only if $d_i + m_i = d_j + m_j$ for all $i, j \in [n]$.

X.Y. Yuan, M. Zhang, M. Lu, Some upper bounds on the eigenvalues of uniform hypergraphs, Linear Algebra Appl. 484

Applications of Theorem 8 to a connected k-uniform hypergraph

Take $R_i = 1$ for each $i \in [n]$. Then $R'_i = d_i$ for any $i \in [n]$.

Corollary 4

Let $\mathcal{H}$ be a connected k-uniform hypergraph on n vertices. Then

$$\min_{e \in E(\mathcal{H})} \min_{\{i,j\} \subseteq e} (d_i + d_j) \leq \rho(\mathbb{Q}(\mathcal{H})) \leq \max_{e \in E(\mathcal{H})} \max_{\{i,j\} \subseteq e} (d_i + d_j). \quad (8)$$

Moreover, one of the equalities in (8) holds if and only if $\mathcal{H}$ is a regular hypergraph.

X.Y. Yuan, M. Zhang, M. Lu, Some upper bounds on the eigenvalues of uniform hypergraphs, Linear Algebra Appl. 484 (2015), 540–549.

Thank You for Your Attention!