Panconnectedness of K-trees with Sufficiently Large Toughness

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Abstract

In this paper, we prove that if the toughness of a k-tree G is at least $(k+1)/3$, then G is panconnected for $k \geq 3$, or G is vertex pancyclic for $k = 2$. This result improves a result of Broersma, Xiong and Yoshimoto.

Keywords: Panconnectedness, toughness, k-tree.

1. Concepts

(1) K-tree: K_k is the smallest k-tree, and a graph G on at least k+1 vertices is a k-tree if and only if it contains a vertex v of degree k such that the neighbours of v are mutually adjacent and $G - v$ is a k-tree. Obviously a 1-tree is just a tree.

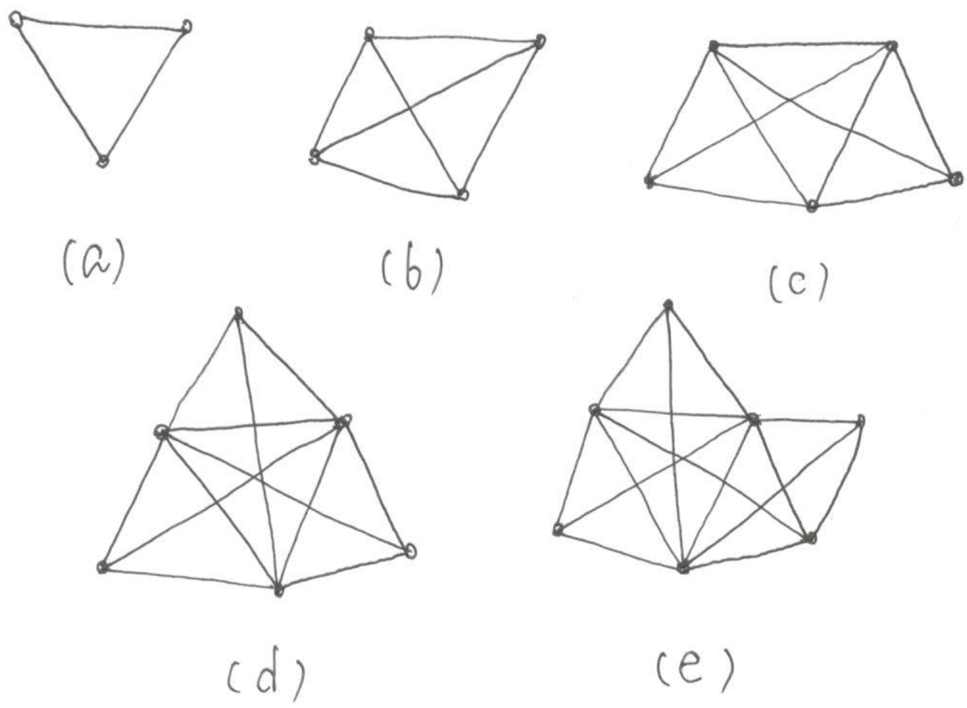

Figure 1 (k-tree, k=3)

(2) Hamiltonicity

(2.1) A graph $G = (V, E)$ is Hamiltonian if G has a cycle of length $v(G)$.

(2.2) A graph G is called pancyclic if there is a cycle C of length L for each integer L from 3 to $v(G)$ in G.

(2.3) A graph G is called vertex pancyclic if, for each vertex v in G, there is a cycle C containing v of length L for each integer L from 3 to $\nu(G)$ in G.

(2.4) A graph G is called edge pancyclic if, for each edge e in G, there is a cycle C containing e of length L for each integer L from 3 to $\nu(G)$ in G.

(2.5) A graph G is called Hamiltonian connected if , for each pair of vertices u and v, there is a Hamiltonian path P from u to v in G.

(2.6) A graph G is called panconnected if, for each pair of vertices u and v in G, there is a path P from u to v of length L for each integer L from d(u, v) to $\nu(G) - 1$ in G.

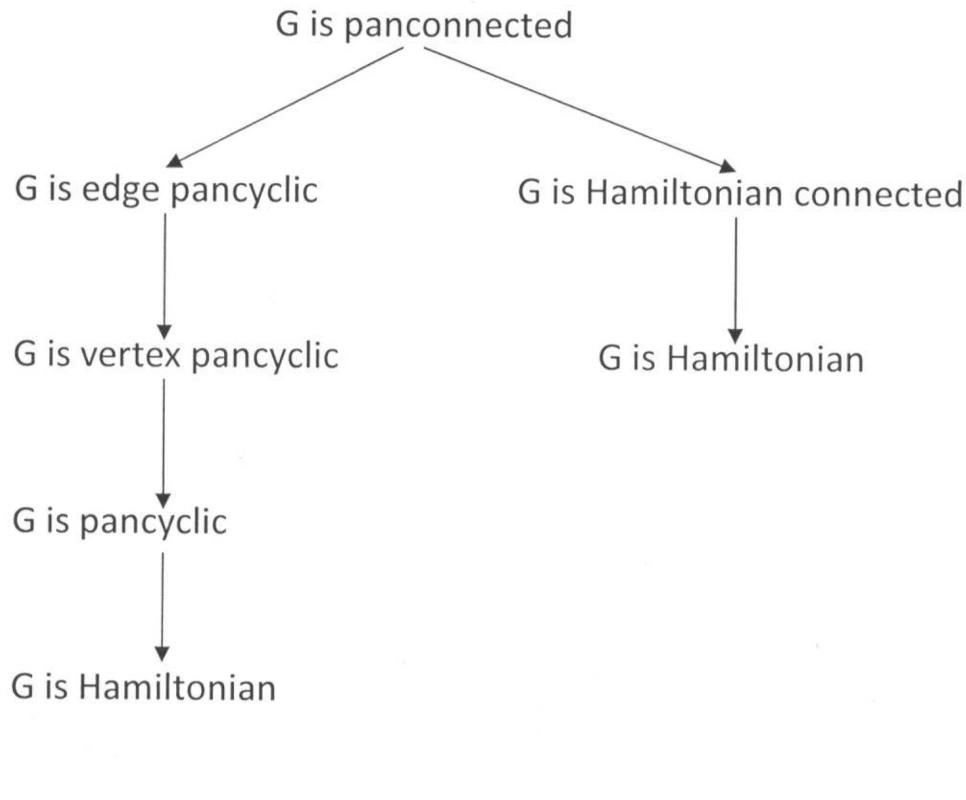

Figure 2

(3) A graph G is called a chordal graph if G contains no chordless cycle of length at least 4.

K-trees ($k \geq 2$) are a subclass of chordal graphs.

(4) A graph G is called t-tough if $|S| \geq t\omega(G-S)$ for each subset S of $V(G)$ with $\omega(G-S) > 1$. The toughness of G is denoted by $\tau(G)$ and is defined as follows: If G is not complete, then

$$\tau(G) = \min\left\{\frac{|S|}{\omega(G-S)}\right\},$$

where the minimum is taken over all cutsets S of vertices in G, otherwise $G = K_n$ and $\tau(G) = \infty$.

2. Introduction

The concept of toughness was introduced by Chvátal in 1973, it is clear that being 1-tough is a necessary condition for a graph to be Hamiltonian. Chvátal conjectured that there exists a finite constant t_0 such that every t_0-tough graph is Hamiltonian. It had been a long standing conjecture for $t_0 = 2$ until Bauer, Broersma and Veldman showed for every $\varepsilon > 0$, there exists a $(\frac{9}{4} - \varepsilon)$-tough nontraceable graph (a graph without Hamiltonian path), which disproved the conjecture. Chvátal obtained $(\frac{3}{2} - \varepsilon)$-tough graphs without a 2-factor for arbitrary $\varepsilon > 0$. These examples are all chordal. Recently Bauer, Katona, Kratsch and Veldman showed that every $\frac{3}{2}$-tough chordal graph has a 2-factor. Motivated by this result, Kratsch raised the question whether every $\frac{3}{2}$-tough chordal graph is Hamiltonian. But Bauer et al. showed that there is an infinite class of chordal graphs with toughness close to $\frac{7}{4}$ having no Hamiltonian path, and hence no Hamiltonian cycle. However, Böhme et al. showed that let G be a chordal planar graph with $\tau(G) > 1$, then G is Hamiltonian. Although $\frac{3}{2}$-tough chordal graphs are not necessarily Hamiltonian, Chen et al. proved that every 18-tough chordal graph is Hamiltonian. Recently, Broersma, Xiong and Yoshimoto showed that k-trees are Hamiltonian if the toughness is at least $\frac{k+1}{3}$ for $k \geq 2$. The authors of this paper try to extend the result of Broersma et al. to panconnectedness.

3. The main results

Theorem 1: If a k-tree G $(k \geq 3)$ has toughness $\tau(G) \geq \frac{k+1}{3}$, then G is panconnected.

Theorem 2: A 1-tough 2-tree G with $\nu(G) \geq 3$ is vertex pancyclic.

A 1-tough 2-tree G is not necessarily edge pancyclic.

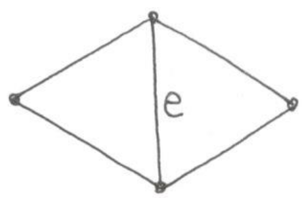

Figure 3