

RAINBOW TRIANGLES IN 3-EDGE-COLORED GRAPHS

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PROBLEM

Find a 3-edge-coloring of a complete graph K_n maximizing the number of copies of rainbow colored triangles



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Color edges randomly, expected density $\frac{2}{9}$.

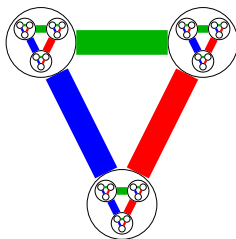
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Find a 3-edge-coloring of a complete graph K_n maximizing the number of copies of rainbow colored triangles



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Iterated blow-up of triangle




$$\frac{1}{4} =$$



denotes graph and/or its density

$F(n) = \max \#$ of  over all 3-edge-colorings of K_n

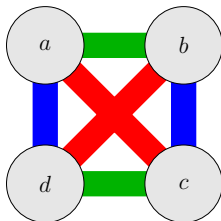
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CONJECTURE (ERDŐS AND SÓS; '72⁻)

For all $n > 0$,

$$F(n) = F(a) + F(b) + F(c) + F(d) + abc + abd + acd + bcd,$$

where $a + b + c + d = n$; a, b, c, d are as equal as possible, and $F(0) = 0$.



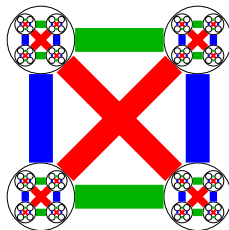
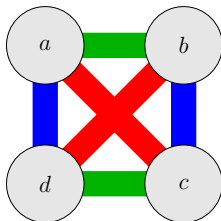
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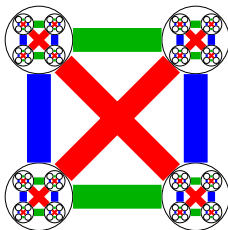
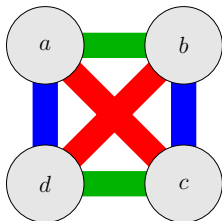
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
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
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
0.4 = 


FLAG ALGEBRAS APPLICATION

Construction: $0.4 \leq$ 


- get a matching upper bound  ≈ 0.4
- round the result
- get subgraphs with 0 density
- get extremal construction (stability)

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
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
Flag algebras (on 6 vertices) give only

 $\leq 0.4006,$


not enough for rounding.

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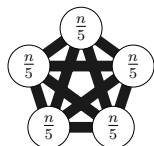
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The iterative extremal construction is causing troubles....

NOT ITERATED EXTREMAL CONSTRUCTIONS

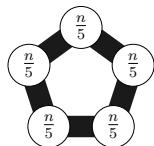
THEOREM (TURÁN)

of edges over K_1 -free graphs is maximized by



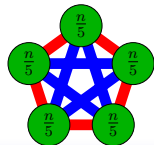
THEOREM (HATAMI, HLADKÝ, KRÁL, NORINE, RAZBOROV)

of C_5 s over triangle-free graphs is maximized by





THEOREM (CUMMINGS, KRÁL, PFENDER, SPERFELD, TREGLOWN, YOUNG)

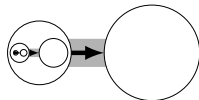
of monochromatic triangles over 3-edge-colored graphs is minimized by




ITERATED EXTREMAL CONSTRUCTIONS

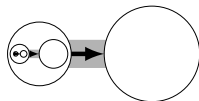
THEOREM (FALGAS-RAVRY, VAUGHAN)

of  and  is maximized by




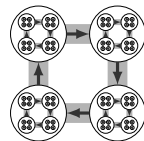
THEOREM (HUANG)

of  is maximized by



THEOREM (HLADKÝ, KRÁL, NORINE)

of  is maximized by



OUR MAIN RESULT

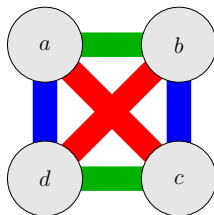
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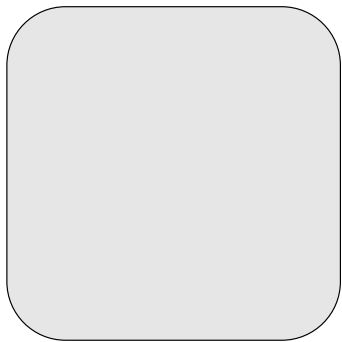
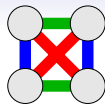


SKETCH OF PROOF

Goal: maximizing



gives edge-coloring like

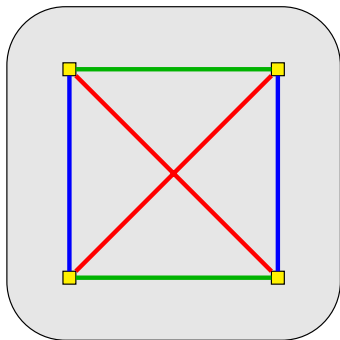
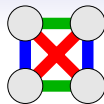


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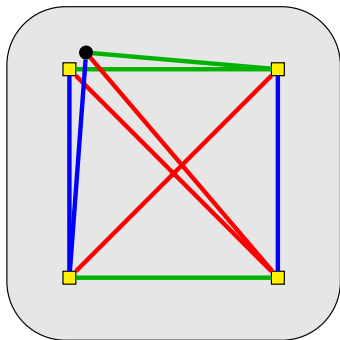
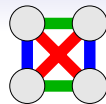
- pick a properly 3-edge-colored K_4

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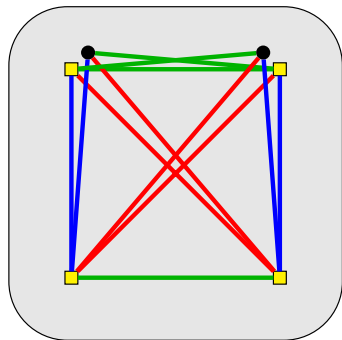
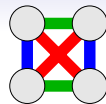
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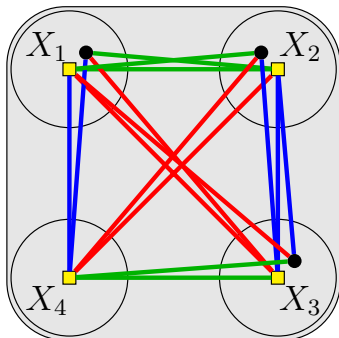
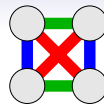
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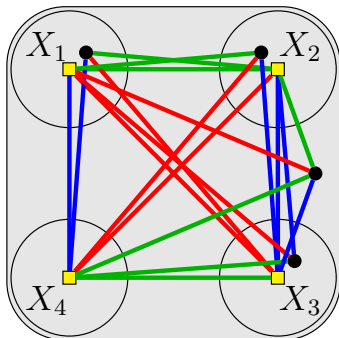
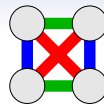
- pick a properly 3-edge-colored K_4
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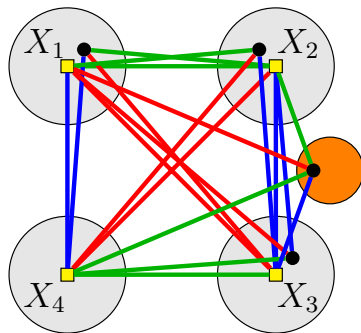
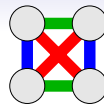
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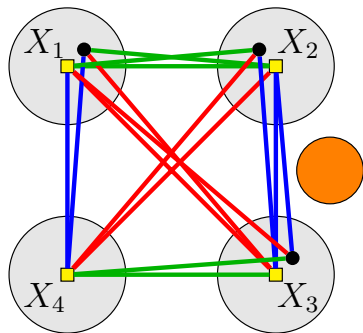
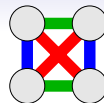
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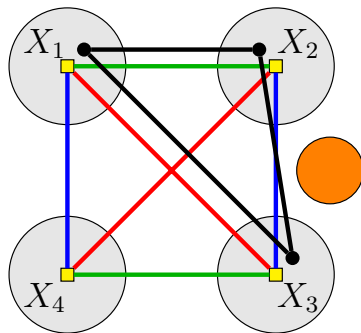
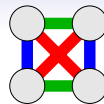
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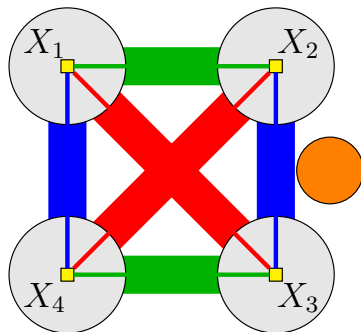
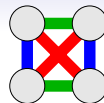
- pick a properly 3-edge-colored K_4
- partition the rest
- correct edges between X_i s

SKETCH OF PROOF

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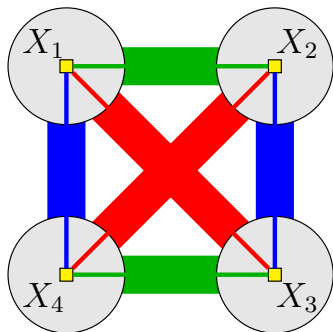
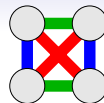
- pick a properly 3-edge-colored K_4
- partition the rest
- correct edges between X_i s
- no orange trash

SKETCH OF PROOF

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gives edge-coloring like



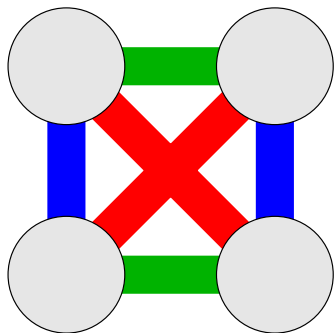
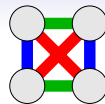
- pick a properly 3-edge-colored K_4
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- balance sizes of X_i s

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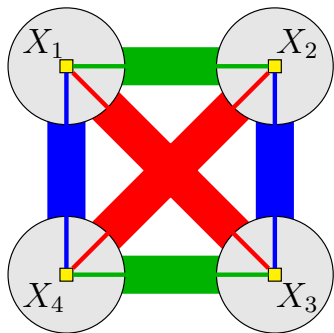
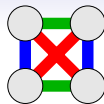
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How to pick the properly 3-edge-colored K_4 ?

($|X_i|$ s close to $0.25n$, few wrongly colored edges, small trash)

HOW TO PICK K_4 ?

Use Flag Algebras!

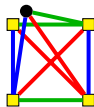
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maximizing



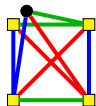
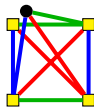
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$$(n - 4) \geq$$

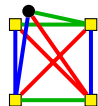
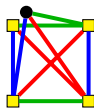
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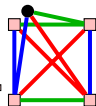
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maximizing



$$(n-4) \geq \frac{1}{\binom{n}{4}} \sum \binom{n-4}{4} (n-4)$$



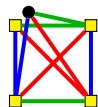
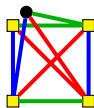
HOW TO PICK K_4 ?

Use Flag Algebras!

Try 1: Pick



maximizing



$(n - 4) \geq$

$$\frac{1}{\text{Diagram of } K_4 \text{ with 2 blue, 2 green, 2 red edges and 2 black dots}} \binom{n}{4}$$

$$\sum \text{Diagram of } K_4 \text{ with 2 blue, 2 green, 2 red edges and 2 black dots}$$

$(n - 4) =$

$$\frac{2 \cdot \text{Diagram of } K_4 \text{ with 2 blue, 2 green, 2 red edges and 2 black dots} \binom{n}{5}}{\text{Diagram of } K_4 \text{ with 2 blue, 2 green, 2 red edges and 2 black dots} \binom{n}{4}}$$

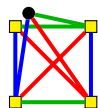
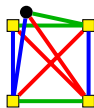
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maximizing



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$$\frac{1}{\text{Diagram of } K_4} \binom{n}{4}$$

$$\sum \text{Diagram of } K_4$$

$(n - 4)$

$$= \frac{2 \cdot \text{Diagram of } K_4 \text{ with black vertex} \binom{n}{5}}{\text{Diagram of } K_4 \binom{n}{4}}$$

$$= \frac{2 \cdot \text{Diagram of } K_4 \text{ with black vertex}}{5 \cdot \text{Diagram of } K_4} (n - 5)$$

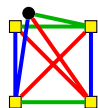
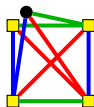
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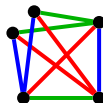


$$(n-4) \geq \frac{1}{\begin{array}{c} \text{diagonal} \\ \text{square} \end{array} \binom{n}{4}} \sum_{\begin{array}{c} \text{diagonal} \\ \text{square} \end{array}} \begin{array}{c} \text{diagonal} \\ \text{square} \end{array} (n-4) = \frac{2 \begin{array}{c} \text{diagonal} \\ \text{square} \end{array} \binom{n}{5}}{\begin{array}{c} \text{diagonal} \\ \text{square} \end{array} \binom{n}{4}} = \frac{2 \begin{array}{c} \text{diagonal} \\ \text{square} \end{array}}{5 \begin{array}{c} \text{diagonal} \\ \text{square} \end{array}} (n-5)$$

FA:



≥ 0.4 then




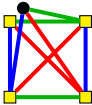
$> 0.23516,$




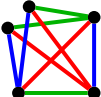

< 0.0952

HOW TO PICK K_4 ?

Use Flag Algebras!


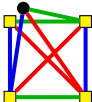
Try 1: Pick  maximizing  > 0.988

$$\begin{array}{c} \text{Diagram: Square with yellow vertices, blue vertical edges, green horizontal edges, red diagonal edges, and a green triangle on the top edge.} \\ (n-4) \geq \frac{1}{\begin{array}{c} \text{Diagram: Square with black vertices, blue vertical edges, green horizontal edges, red diagonal edges.} \\ \binom{n}{4} \end{array}} \sum_{\begin{array}{c} \text{Diagram: Square with black vertices, blue vertical edges, green horizontal edges, red diagonal edges, and a green triangle on the bottom edge.} \\ \text{Diagram: Square with black vertices, blue vertical edges, green horizontal edges, red diagonal edges.} \end{array}} (n-4) = \frac{2 \begin{array}{c} \text{Diagram: Square with black vertices, blue vertical edges, green horizontal edges, red diagonal edges, and a green triangle on the top edge.} \\ \binom{n}{5} \end{array}}{\begin{array}{c} \text{Diagram: Square with black vertices, blue vertical edges, green horizontal edges, red diagonal edges.} \\ \binom{n}{4} \end{array}} = \frac{2 \begin{array}{c} \text{Diagram: Square with black vertices, blue vertical edges, green horizontal edges, red diagonal edges, and a green triangle on the top edge.} \\ \binom{n}{5} \end{array}}{5 \begin{array}{c} \text{Diagram: Square with black vertices, blue vertical edges, green horizontal edges, red diagonal edges.} \\ \binom{n}{4} \end{array}} (n-5)$$


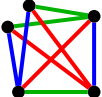

FA:  ≥ 0.4 then  > 0.23516 ,  < 0.0952

HOW TO PICK K_4 ?

Use Flag Algebras!

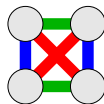
Try 1: Pick  maximizing  > 0.988

$$\begin{array}{c} \text{Diagram} \\ (n-4) \end{array} \geq \frac{1}{\begin{array}{c} \text{Diagram} \\ \binom{n}{4} \end{array}} \sum_{\begin{array}{c} \text{Diagram} \\ \binom{n}{4} \end{array}} \begin{array}{c} \text{Diagram} \\ (n-4) \end{array} = \frac{2 \begin{array}{c} \text{Diagram} \\ \binom{n}{5} \end{array}}{\begin{array}{c} \text{Diagram} \\ \binom{n}{4} \end{array}} = \frac{2 \begin{array}{c} \text{Diagram} \\ \binom{n}{5} \end{array}}{5 \begin{array}{c} \text{Diagram} \\ \binom{n}{4} \end{array}} (n-5)$$

FA:  ≥ 0.4 then  > 0.23516 ,  < 0.0952


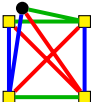
Result for K_n :

$$|X_1| + |X_2| + |X_3| + |X_4| > 0.988(n-4)$$



HOW TO PICK K_4 ?

Use Flag Algebras!

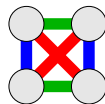
Try 1: Pick  maximizing  > 0.988

$$\begin{array}{c} \text{Diagram: Square K4 with yellow vertices, top/bottom edges green, left/right edges and diagonals blue. An additional black vertex is connected to the top-left and top-right vertices by green edges.} \\ (n-4) \geq \frac{1}{\begin{array}{c} \text{Diagram: Square K4 with yellow vertices, top/bottom edges green, left/right edges and diagonals blue.} \\ \binom{n}{4} \end{array}} \sum_{\begin{array}{c} \text{Diagram: Square K4 with yellow vertices, top/bottom edges green, left/right edges and diagonals blue.} \\ \text{Diagram: Square K4 with yellow vertices, top/bottom edges green, left/right edges and diagonals blue.} \end{array}} (n-4) = \frac{2 \cdot \begin{array}{c} \text{Diagram: Square K4 with yellow vertices, top/bottom edges green, left/right edges and diagonals blue. An additional black vertex is connected to the top-left and top-right vertices by green edges.} \\ \binom{n}{5} \end{array}}{\begin{array}{c} \text{Diagram: Square K4 with yellow vertices, top/bottom edges green, left/right edges and diagonals blue.} \\ \binom{n}{4} \end{array}} = \frac{2 \cdot \begin{array}{c} \text{Diagram: Square K4 with yellow vertices, top/bottom edges green, left/right edges and diagonals blue. An additional black vertex is connected to the top-left and top-right vertices by green edges.} \\ \binom{n}{5} \end{array}}{5 \cdot \begin{array}{c} \text{Diagram: Square K4 with yellow vertices, top/bottom edges green, left/right edges and diagonals blue.} \\ \binom{n}{4} \end{array}} (n-5)$$

$$\text{FA: } \begin{array}{c} \text{Diagram: Triangle with black vertices. Left edge blue, right edge red, bottom edge green.} \end{array} \geq 0.4 \text{ then } \begin{array}{c} \text{Diagram: Square K4 with black vertices. Top/bottom edges green, left/right edges and diagonals blue. An additional black vertex is connected to the top-left and top-right vertices by green edges.} \end{array} > 0.23516, \quad \begin{array}{c} \text{Diagram: Square K4 with black vertices. Top/bottom edges green, left/right edges and diagonals blue.} \end{array} < 0.0952$$

Result for K_n :

$$|X_1| + |X_2| + |X_3| + |X_4| > 0.988(n-4)$$



Balancing needed...

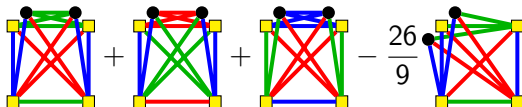
HOW TO PICK K_4 ?

Use Flag Algebras!

Try 2: Pick




maximizing



HOW TO PICK K_4 ?

Use Flag Algebras!

Try 2: Pick  maximizing


$$\begin{array}{c}
 \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} - \frac{26}{9} \text{Diagram 4}
 \end{array}$$

The diagrams are square graphs with four yellow vertices and four black vertices (top and bottom). The edges are colored red, green, or blue. Diagram 1: Red diagonals, green top/bottom, blue left/right. Diagram 2: Red top/bottom, green left/right, blue diagonals. Diagram 3: Red left/right, green top/bottom, blue diagonals. Diagram 4: Red top/bottom, green left/right, blue diagonals, with an additional red edge between the two black vertices on the left.

$$\text{FA: } \frac{4}{15} \left(\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} \right) - \frac{26}{45} \text{Diagram 4} > 0.002629$$

HOW TO PICK K_4 ?

Use Flag Algebras!

Try 2: Pick  maximizing

$$\begin{array}{c} \text{Diagram 1} \end{array} + \begin{array}{c} \text{Diagram 2} \end{array} + \begin{array}{c} \text{Diagram 3} \end{array} - \frac{26}{9} \begin{array}{c} \text{Diagram 4} \end{array} > 0.0276$$


The diagrams are 4x4 bipartite graphs with yellow vertices on the left and right, and black vertices on the top and bottom. Edges are colored red, green, or blue. Diagram 1: Red edges are all possible between top and bottom black vertices. Green edges are all possible between top and bottom yellow vertices. Blue edges are all possible between top and bottom black vertices. Diagram 2: Red edges are all possible between top and bottom black vertices. Green edges are all possible between top and bottom yellow vertices. Blue edges are all possible between top and bottom black vertices. Diagram 3: Red edges are all possible between top and bottom black vertices. Green edges are all possible between top and bottom yellow vertices. Blue edges are all possible between top and bottom black vertices. Diagram 4: Red edges are all possible between top and bottom black vertices. Green edges are all possible between top and bottom yellow vertices. Blue edges are all possible between top and bottom black vertices.

$$\text{FA: } \frac{4}{15} \left(\begin{array}{c} \text{Diagram 1} \end{array} + \begin{array}{c} \text{Diagram 2} \end{array} + \begin{array}{c} \text{Diagram 3} \end{array} \right) - \frac{26}{45} \begin{array}{c} \text{Diagram 4} \end{array} > 0.002629$$

The diagrams are 4x4 bipartite graphs with yellow vertices on the left and right, and black vertices on the top and bottom. Edges are colored red, green, or blue. Diagram 1: Red edges are all possible between top and bottom black vertices. Green edges are all possible between top and bottom yellow vertices. Blue edges are all possible between top and bottom black vertices. Diagram 2: Red edges are all possible between top and bottom black vertices. Green edges are all possible between top and bottom yellow vertices. Blue edges are all possible between top and bottom black vertices. Diagram 3: Red edges are all possible between top and bottom black vertices. Green edges are all possible between top and bottom yellow vertices. Blue edges are all possible between top and bottom black vertices. Diagram 4: Red edges are all possible between top and bottom black vertices. Green edges are all possible between top and bottom yellow vertices. Blue edges are all possible between top and bottom black vertices.

HOW TO PICK K_4 ?

Use Flag Algebras!

Try 2: Pick  maximizing

$$\begin{array}{c} \text{Diagram 1} \end{array} + \begin{array}{c} \text{Diagram 2} \end{array} + \begin{array}{c} \text{Diagram 3} \end{array} - \frac{26}{9} \begin{array}{c} \text{Diagram 4} \end{array} > 0.0276$$

The diagrams are 4x4 graphs with vertices colored yellow (corners) and black (center). Edges are colored red, green, or blue. Diagram 1: Red edges are diagonals and horizontal edges; green edges are vertical edges; blue edges are outer edges. Diagram 2: Red edges are diagonals and vertical edges; green edges are horizontal edges; blue edges are outer edges. Diagram 3: Red edges are diagonals and vertical edges; green edges are horizontal edges; blue edges are outer edges. Diagram 4: Red edges are diagonals and horizontal edges; green edges are vertical edges; blue edges are outer edges.

$$\text{FA: } \frac{4}{15} \left(\begin{array}{c} \text{Diagram 1} \end{array} + \begin{array}{c} \text{Diagram 2} \end{array} + \begin{array}{c} \text{Diagram 3} \end{array} \right) - \frac{26}{45} \begin{array}{c} \text{Diagram 4} \end{array} > 0.002629$$

Final equation:

$$2 \sum_{1 \leq i < j \leq 4} |X_i| |X_j| - |F| - \frac{26}{9} \sum_{1 \leq i \leq 4} |X_i|^2 > 0.0276 n^2$$

F = wrongly colored edges.

HOW THE FIRST STEP WORKED

Quadratic programming

$$2 \sum_{1 \leq i < j \leq 4} |X_i||X_j| - |F| - \frac{26}{9} \sum_{1 \leq i \leq 4} |X_i|^2 > 0.0276n^2$$

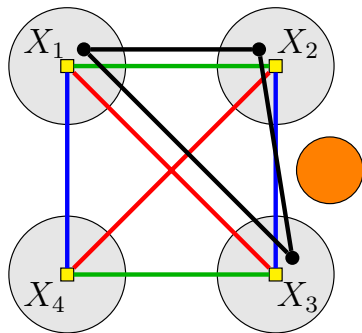
Use Lagrange multipliers:

$$0.244n < |X_i| < 0.256n$$

$$|\text{Trash}| < 0.006n$$

$$|F| < 0.00008 \binom{n}{2}$$

F = wrongly colored edges.



MAIN RESULT

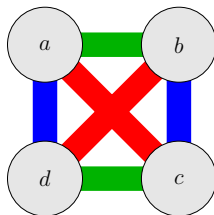
$F(n) = \max \#$ of  over all coloring of K_n

THEOREM (BALOGH, H., LIDICKÝ, PFENDER, VOLEC, YOUNG 2017)

For all $n > n_0$,

$$F(n) = F(a) + F(b) + F(c) + F(d) + abc + abd + acd + bcd,$$

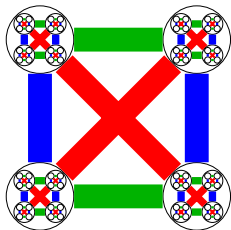
where $a + b + c + d = n$; a, b, c, d are as equal as possible.



MORE RESULTS

THEOREM

*# of rainbow K_3 s is
maximized by*

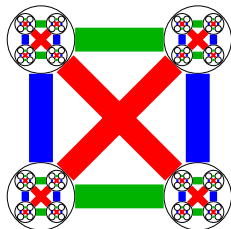


if on 4^k vertices.

MORE RESULTS

THEOREM

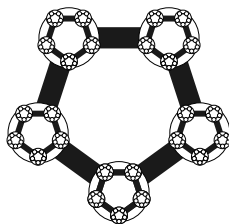
of rainbow K_3 s is maximized by



if on 4^k vertices.

THEOREM

of induced C_5 s is maximized by

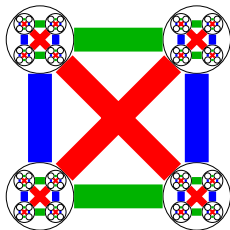


if on 5^k vertices.

MORE RESULTS

THEOREM

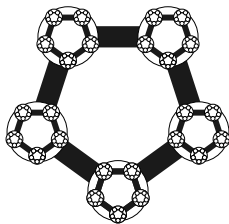
of rainbow K_3 s is maximized by



if on 4^k vertices.

THEOREM

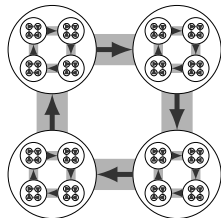
of induced C_5 s is maximized by



if on 5^k vertices.

THEOREM

of induced oriented C_4 s is maximized by

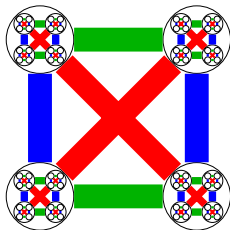


if on 4^k vertices.

MORE RESULTS

THEOREM

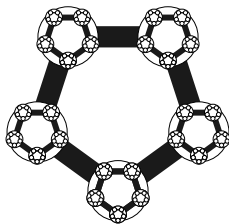
of rainbow K_3 s is maximized by



if on 4^k vertices.

THEOREM

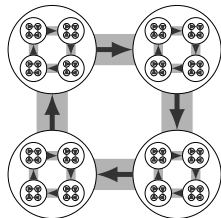
of induced C_5 s is maximized by



if on 5^k vertices.

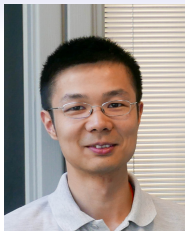
THEOREM

of induced oriented C_4 s is maximized by



if on 4^k vertices.

(for all k)



Thank you for listening!

