Strong Subgraph k-Connectivity of Digraphs

Yuefang Sun

joint work with Gregory Gutin, Anders Yeo, Xiaoyan Zhang

yuefangsun2013@163.com
Department of Mathematics
Shaoxing University, China

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Outline

Introduction

Algorithms and Complexity

Sharp Bounds and Characterizations

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Connectivity

For any two distinct vertices x and y in G, the *local connectivity* $\kappa_G(x, y)$ is the maximum number of internally disjoint paths connecting x and y.

Then the connectivity of G is defined as

$$\kappa(G) = \min\{\kappa_G(x, y) \mid x, y \in V(G), x \neq y\}$$

S-tree

For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, an *S-Steiner tree* or a *Steiner tree connecting S* (or simply, an S-tree) is a such subgraph T of G that is a tree with $S \subseteq V(T)$.

Two edge-disjoint S-trees T_1 and T_2 are said to be internally disjoint if $V(T_1) \cap V(T_2) = S$ and $E(T_1) \cap E(T_2) = \emptyset$.

Generalized k-connectivity

Generalized local connectivity $\kappa_G(S)$ is the maximum number of internally disjoint S-trees in G.

For an integer k with $2 \leq k \leq n$, the generalized k-connectivity (or k-tree-connectivity) is defined as

$$\kappa_k(G) = \min\{\kappa_G(S) \mid S \subseteq V(G), |S| = k\}.$$

Thus, $\kappa_k(G)$ is the minimum value of $\kappa_G(S)$ when S runs over all k-subsets of $V(G)$. (Hager, JCTB 1985)

X. Li and Y. Mao, Generalized Connectivity of Graphs, Springer, Switzerland, 2016.

Two extremal cases

For $k = 2$, $\kappa_2(G) = \kappa(G)$.

For $k = n$, $\kappa_n(G) = STP(G)$, where $STP(G)$ is the spanning tree packing number of G, that is, the maximum number of edge-disjoint spanning trees contained in G.

For the spanning tree packing number, see the following two surveys:

K. Ozeki, T. Yamashita, Spanning trees: A survey, Graphs Combin. 27(1)(2011), 1–26.

E. Palmer, On the spanning tree packing number of a graph: a survey, Discrete Math. 230(2001), 13–21.

S-strong subgraph

Let $D = (V(D), A(D))$ be a digraph of order n, $S \subseteq V$ a k-subset of $V(D)$ and $2 \leq k \leq n$.

Strong subgraphs $D_1, \ldots, D_p$ containing S (S-strong subgraph) are said to be internally disjoint if $V(D_i) \cap V(D_j) = S$ and $A(D_i) \cap A(D_j) = \emptyset$ for all $1 \leq i < j \leq p$.

Strong subgraph k-connectivity

Let $\kappa_S(D)$ be the maximum number of internally disjoint strong digraphs containing S in D. The strong subgraph k-connectivity is defined as

$$\kappa_k(D) = \min\{\kappa_S(D) \mid S \subseteq V(D), |S| = k\}.$$

(Sun, Gutin, Yeo, Zhang, 2018+)

For $k = 2$, $\kappa_2(\overleftrightarrow{G}) = \kappa(G)$.

For $k = n$, $\kappa_n(D)$ is the maximum number of arc-disjoint spanning strong subgraph of D. Hence it relates to the subdigraph packing problem, see

J. Bang-Jensen, M. Kriesell, Electronic Notes in Discrete Mathematics 34 (2009) 179–183.

J. Bang-Jensen, A. Yeo, Combinatorica, 24 (3) (2004) 331–349.

J. Bang-Jensen, J. Huang, J. Combin. Theory Ser. B, 102 (2012) 701–714.

J. Bang-Jensen, A. Yeo, Theoret. Comput. Sci., 438 (2012) 48–54.

Outline

A general digraph

Theorem

Let $k \geq 2$ and $\ell \geq 2$ be fixed integers. Let D be a digraph and $S \subseteq V(D)$ with $|S| = k$. The problem of deciding whether $\kappa_S(D) \geq \ell$ is NP-complete.

(Sun, Gutin, Yeo, Zhang, 2018+)

Directed q-Linkage problem

For a fixed integer $q \geq 2$, given a digraph D and a (terminal) sequence $(s_1, t_1, \ldots, s_q, t_q)$ of distinct vertices of D, decide whether D has q vertex-disjoint paths $P_1, \ldots, P_q$, where P_i starts at s_i and ends at t_i for all $i \in [q]$.

Theorem

The directed 2-linkage problem is NP-complete.

(Fortune, Hopcroft, Wyllie, TCS 1980)

Semicomplete digraph

A digraph is semicomplete if there is at least one arc between any pair of vertices.

Theorem

For any fixed integers $k, \ell \geq 2$, we can decide whether $\kappa_k(D) \geq \ell$ for a semicomplete digraph D in polynomial time.

(Sun, Gutin, Yeo, Zhang, 2018+)

A result by Chudnovsky, Scott and Seymour

Theorem

Let q and c be fixed positive integers. Then the DIRECTED q-LINKAGE *problem on a digraph D whose vertex set can be partitioned into c sets each inducing a semicomplete digraph and a terminal sequence $(s_1, t_1, \ldots, s_q, t_q)$ of distinct vertices of D, can be solved in polynomial time.*

(Chudnovsky, Scott and Seymour, 2018+)

Symmetric digraph

A digraph D is called symmetric if for every arc xy there is an opposite arc yx. Thus, a symmetric digraph D can be obtained from its underlying undirected graph G by replacing each edge of G with the corresponding arcs of both directions. We say that D is the complete biorientation of G and denote this by $D = \overleftrightarrow{G}$.

Theorem

For every graph G we have $\kappa_2(\overleftrightarrow{G}) = \kappa(G)$.

(Sun, Gutin, Yeo, Zhang, 2018+)

Corollary

For a graph G, $\kappa_2(\overleftrightarrow{G})$ can be computed in polynomial time.

Symmetric digraph

Theorem

For any fixed integer $k \geq 3$, given a symmetric digraph D, a k-subset S of $V(D)$ and an integer ℓ ($\ell \geq 1$), deciding whether $\kappa_S(D) \geq \ell$, is NP-complete.

(Sun, Gutin, Yeo, Zhang, 2018+)

This theorem assumes that k is fixed but ℓ is a part of input. When both k and ℓ are fixed, the problem of deciding whether $\kappa_S(D) \geq \ell$ for a symmetric digraph D, is polynomial-time solvable.

The CLLM problem

Given a tripartite graph $G = (V, E)$ with a 3-partition $(\overline{U}, \overline{V}, \overline{W})$ such that $|\overline{U}| = |\overline{V}| = |\overline{W}| = q$, decide whether there is a partition of V into q disjoint 3-sets $V_1, \ldots, V_q$ such that for every $V_i = \{v_{i_1}, v_{i_2}, v_{i_3}\}$ $v_{i_1} \in \overline{U}, v_{i_2} \in \overline{V}, v_{i_3} \in \overline{W}$ and $G[V_i]$ is connected.

Lemma

The CLLM Problem is NP-complete.

(Chen, Li, Liu and Mao, JOCO 2017)

Symmetric digraph

Theorem

Let $k, \ell \geq 2$ be fixed. For any symmetric digraph D and $S \subseteq V(D)$ with $|S| = k$ we can in polynomial time decide whether $\kappa_S(D) \geq \ell$.

(Sun, Gutin, Yeo, Zhang, 2018+)

A lemma

Lemma

Let $k, \ell \geq 2$ be fixed. Let G be a graph and let $S \subseteq V(G)$ be an independent set in G with $|S| = k$. For $i \in [\ell]$, let D_i be any set of arcs with both end-vertices in S. Let a forest F_i in G be called (S, D_i)-acceptable if the digraph $\overleftrightarrow{F_i} + D_i$ is strong and contains S. In polynomial time, we can decide whether there exists edge-disjoint forests $F_1, F_2, \ldots, F_\ell$ such that F_i is (S, D_i)-acceptable for all $i \in [\ell]$ and $V(F_i) \cap V(F_j) \subseteq S$ for all $1 \leq i < j \leq \ell$.

(Sun, Gutin, Yeo, Zhang, 2018+)

Outline

Two observations

Observation

If D' is a strong spanning digraph of a strong digraph D, then $\kappa_k(D') \leq \kappa_k(D)$.

Observation

For all digraphs D and $k \geq 2$ we have $\kappa_k(D) \leq \delta^+(D)$ and $\kappa_k(D) \leq \delta^-(D)$.

Tillson's decomposition theorem

Theorem

The arcs of $\overleftrightarrow{K}_n$ can be decomposed into Hamiltonian cycles if and only if $n \neq 4, 6$.

(Tillson, JCTB 1980)

Complete digraph

Lemma

For $2 \leq k \leq n$, we have

$$\kappa_k(\overleftrightarrow{K}_n) = \begin{cases} n-1, & \text{if } 2 \leq k \leq n \text{ and } k \notin \{4,6\}; \\ n-2, & \text{otherwise.} \end{cases}$$

(Sun, Gutin, Yeo, Zhang, 2018+)

Sharp bounds

Theorem

Let $2 \leq k \leq n$. For a strong digraph D of order n, we have

$$1 \leq \kappa_k(D) \leq n-1.$$

Moreover, both bounds are sharp, and the upper bound holds if and only if $D \cong \overleftrightarrow{K}_n$, $2 \leq k \leq n$ and $k \notin \{4, 6\}$.

(Sun, Gutin, Yeo, Zhang, 2018+)

An improved upper bound

Theorem

For $k \in \{2, \ldots, n\}$ and $n \geq \kappa(D) + k$, we have

$$\kappa_k(D) \leq \kappa(D).$$

Moreover, the bound is sharp.

(Sun & Gutin, 2018+)

Minimally strong subgraph (k, ℓ)-connected digraph

A digraph $D = (V(D), A(D))$ is called minimally strong subgraph (k, ℓ)-connected if $\kappa_k(D) \geq \ell$ but for any arc $e \in A(D)$, $\kappa_k(D - e) \leq \ell - 1$.

Let $\mathfrak{F}(n, k, \ell)$ be the set of all minimally strong subgraph (k, ℓ)-connected digraphs with order n. We define

$$F(n, k, \ell) = \max\{|A(D)| \mid D \in \mathfrak{F}(n, k, \ell)\}$$

and

$$f(n, k, \ell) = \min\{|A(D)| \mid D \in \mathfrak{F}(n, k, \ell)\}.$$

We further define

$$Ex(n, k, \ell) = \{D \mid D \in \mathfrak{F}(n, k, \ell), |A(D)| = F(n, k, \ell)\}$$

and

$$ex(n, k, \ell) = \{D \mid D \in \mathfrak{F}(n, k, \ell), |A(D)| = f(n, k, \ell)\}.$$

Proposition

The following assertions hold:

(i) A digraph D is minimally strong subgraph $(k, 1)$-connected if and only if D is minimally strong digraph;

(ii) For $k \neq 4, 6$, a digraph D is minimally strong subgraph $(k, n-1)$-connected if and only if $D \cong \overleftrightarrow{K}_n$.

(Sun & Gutin, 2018+)

The case $k = 2, \ell = n - 2$

Theorem

A digraph D is minimally strong subgraph $(2, n-2)$-connected if and only if D is a digraph obtained from the complete digraph $\overleftrightarrow{K}_n$ by deleting an arc set M such that $\overleftrightarrow{K}_n[M]$ is a 3-cycle or a union of $\lfloor n/2 \rfloor$ vertex-disjoint 2-cycles. In particular, we have $f(n, 2, n-2) = n(n-1) - 2\lfloor n/2 \rfloor$, $F(n, 2, n-2) = n(n-1) - 3$.

(Sun & Gutin, 2018+)

Note that this theorem implies that $Ex(n, 2, n-2) = \{\overleftrightarrow{K}_n - M\}$ where M is an arc set such that $\overleftrightarrow{K}_n[M]$ is a directed 3-cycle, and $ex(n, 2, n-1) = \{\overleftrightarrow{K}_n - M\}$ where M is an arc set such that $\overleftrightarrow{K}_n[M]$ is a union of $\lfloor n/2 \rfloor$ vertex-disjoint directed 2-cycles.

$f(n, k, \ell)$

Theorem

For $2 \leq k \leq n$, we have

$$f(n, k, \ell) \geq n\ell.$$

Moreover, the following assertions hold:

(i) If $\ell = 1$, then $f(n, k, \ell) = n$;
(ii) If $2 \leq \ell \leq n-1$, then $f(n, n, \ell) = n\ell$ for $k = n \notin \{4, 6\}$;
(iii) If n is even and $\ell = n-2$, then $f(n, 2, \ell) = n\ell$.

(Sun & Gutin, 2018+)

$F(n, k, \ell)$

Proposition

We have (i) $F(n, n, \ell) \leq 2\ell(n-1)$; (ii) For every k $(2 \leq k \leq n)$, $F(n, k, 1) = 2(n-1)$ and $Ex(n, k, 1)$ consists of symmetric digraphs whose underlying undirected graphs are trees.

(Sun & Gutin, 2018+)

Outline

Open problems

Conjecture

It is NP-complete to decide for fixed integers $k \geq 2$ and $\ell \geq 2$ and a given digraph D whether $\kappa_k(D) \geq \ell$.

Open problems

The DIRECTED q-LINKAGE problem is polynomial-time solvable for planar digraphs (Schrijver, SIAM J. Comput. 1994) and digraphs of bounded directed treewidth (Johnson, Robertson, Seymour and Thomas, JCTB 2001).

However, we cannot use our approach in this paper directly as the structure of minimum-size strong subgraphs in these two classes of digraphs is more complicated than in semicomplete digraphs.

Certainly, we cannot exclude the possibility that computing strong subgraph k-connectivity in planar digraphs and/or in digraphs of bounded directed treewidth is NP-complete.

Open problems

It would be interesting to determine $f(n,k,n-2)$ and $F(n,k,n-2)$ for every value of $k \geq 3$. (Obtaining characterizations of all $(k,n-2)$-connected digraphs for $k \geq 3$ seems a very difficult problem.)

It would also be interesting to find a sharp upper bound for $F(n,k,\ell)$ for all $k \geq 2$ and $\ell \geq 2$.

Thanks for your attention!