Supereulerian Width of Dense Graphs

Joint work with HJ.Lai,Wei Xiong, Jinquan Xu, Zhengke Miao.

Motivations

- **Menger's Theorem** A graph G is k-edge-connected if and only if for any $u, v \in V(G)$, there exists k edge disjoint (u, v)-path.

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- **Question** How about we replace k edge disjoint path structure in Menger's Theorem by other graph structures?

Definitions

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- **An Eulerian Connected Graph (Luo et al):** A graph G is eulerian connected if for any $u, v \in V(G)$ there is a spanning edge disjoint (u,v)-trail.

Definitions

- **An** $(s; u, v)$**-trail-system:** For a graph G and an integer $s > 0$ and for $u, v \in V(G)$ with $u \neq v$, an $(s; u, v)$-trail-system of G is a subgraph H consisting of s edge-disjoint (u, v)-trails.

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- **The Supereulerian Width** $\mu'(G)$**:** $\mu'(G)$ of a graph G is the largest integer s such that for any $u, v \in V(G)$, G has a spanning $(k; u, v)$-trail-system, for any integer k with $1 \leq k \leq s$.

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- **The Supereulerian Width $\mu'(G)$:** $\mu'(G)$ of a graph G is the largest integer s such that for any $u, v \in V(G)$, G has a spanning $(k; u, v)$-trail-system, for any integer k with $1 \leq k \leq s$.
- Particularly, G is supereulerian if and only if $\mu'(G) \geq 2$.

Existing results

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- **Theorem (Li et al)** Let G be a simple graph on n vertices. For any positive integers p and s with $p \geq 2$, there exists an integer $n = N(s, p)$ and a finite family F_0 of graphs with supereulerian width at most s such that if $\delta(G) \geq \frac{n}{p} - 1$, then either $\mu'(G) \geq s + 1$, or G is contractible to a member in F_0.

Question

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- **Question 2** Can the form of the bound $\frac{n}{p} - 1$ be further generalized?

Our Results

- **Theorem 1.** For any real numbers a, b with $0 < a < 1$ and any integer $s > 0$, there exists a finite family $F = (a, b, s)$ such that for any simple graph G with $n = |V(G)|$, if for any pair of nonadjacent vertices u and v, $\max\{d_G(u), d_G(v)\} \geq an + b$, then $\mu'(G) \geq s + 1$ if and only if G is not contractible to a member in F.

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- **Theorem 2.** For a simple graph G with $|V(G)| = n \geq 141$ and $\kappa'(G) \geq 3$, if for any pair of nonadjacent vertices u and v, $\max\{d_G(u), d_G(v)\} \geq \frac{n}{4} - \frac{3}{2}$, then $\mu'(G) \geq 3$ if and only if G is not contractible to $K_{3,3}$.

Preliminary

- **Lemma 1.** (Li et al) Let $s > 0$ be an integer. Each of the following holds.
  (i) $\mu'(K_1) \geq s+1$.
  (ii) If $e \in E(G)$, then $\mu'(G/e) \geq \mu'(G)$. In particular, if $\mu'(G) \geq s+1$ and $e \in E(G)$, then $\mu'(G/e) \geq s+1$.

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- **Lemma 2.** (P. Li et al) Let G be graph on $n \leq 6$ vertices. Then $\mu'(G) \geq 3$ if and only if $G \ncong K_{3,3}$.

Definitions

- ***s*-collapsible:** A graph G is s-**collapsible** (denoted by $G \in C_s$) if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph Γ_R such that both $O(\Gamma_R) = R$ and $\kappa'(G - E(\Gamma_R)) \geq s - 1$.

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- C_s-**reduced** A graph is C_s-**reduced** if it contains no nontrivial subgraph in C_s. A C_s-reduced graph G' is the the graph obtained from G by contracting each of the maximal C_s subgraphs of G. G' is well defined.

Preliminary

- **Lemma 3.** Let G be a graph with $|V(G)| = n \geq 138$ and $\kappa'(G) \geq 3$ and let G' be the C_2-reduced graph of G. If for any $u, v \in V(G)$ with $uv \notin E(G)$,

$$\max\{d_G(u), d_G(v)\} \geq \frac{n}{4} - \frac{3}{2}$$

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- **Lemma 4.** Let H be a graph with $\kappa'(H) \geq 3$ and $|V(H)| = 7$. If H contains a subgraph $L \cong K_4$, then for any distinct $u, v \in H$ there exists a (u, v)-path P in H such that $H - E(P)$ is 1-collapsible.

proof of Theorem 2

- **Proof. Necessity:** Let G be a graph which is contractible to $K_{3,3}$, then by Lemma 2 and Lemma 1 (ii), $\mu'(G) \leq 2$.

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- **Sufficiency:** Let G be a graph which is not contractible to $K_{3,3}$. Let G' be the $_2$-reduction of G. Then by Lemma 3, $|V(G')| \leq 7$. If $|V(G')| \leq 6$, then since G is not contractible to $K_{3,3}$ and by Lemma 2 we have $\mu'(G') \geq 3$. If $|V(G')| = 7$, then by Lemma 4 we have $\mu'(G') \geq 3$. Finally, since $\mu'(G') \geq 3$, by Lemma 1 we know that $\mu'(G) \geq 3$.

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- **Thank you for your attention!**